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THE

MECHANICAL PRINCIPLES

OF

ENGINEERING

AND

ARCHITECTURE.

BY

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PREFACE.

In the following work, I have proposed to myself to apply the principles of mechanics to the discussion of the most important and obvious of those questions which present themselves in the practice of the engineer and the architect; and I have sought to include in that discussion all the circumstances on which the practical solution of such questions may be assumed to depend. It includes the substance of a course of lectures delivered to the students of King's College in the department of engineering and architecture, during the years 1840, 1841, 1842.*

In the first part I have treated of those portions of the science of Statics which have their application in the theory of machines and the theory of construction.

In the second, of the science of Dynamics, and, under this head, particularly of that union of a continued pressure with a continued motion which has received from English writers the various names of

* The first 170 pages of the work were printed for the use of my pupils in the year 1840. Copies of them were about the same time in the possession of several of my friends in the Universities.

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feet in the projection of the space described*, upon the direction of the pressure; that is, by the product of the pressure by its virtual velocity. Thus, then, we conclude, at once, by the principle of virtual velocities, that if a machine work under a constant equilibrium of the pressures applied to it, or if it work uniformly, then is the aggregate work of those pressures which tend to accelerate its motion equal to the aggregate work of those which tend to retard it; and, by the principle of *vis viva*, that if the machine do not work under an equilibrium of the forces impressed upon it, then is the aggregate work of those which tend to accelerate the motion of the machine greater or less than the aggregate work of those which tend to retard its motion by one half the aggregate of the *vires vivæ* acquired or lost by the moving parts of the system, whilst the work is being done upon it. In no respect have the labours of the illustrious president of the Academy of Sciences more contributed to the developement of the theory of machines than in the application which he has so successfully made to it of this principle of *vis viva.*† In the elementary discussion of this principle, which is given by M. Poncelet, in the introduction to his *Mécanique Industrielle*, he has revived the term *vis inertiæ* (*vis*

* If the direction of the pressure remain always parallel to itself, the space described may be any finite space; if it do not, the space is understood to be so small, that the direction of the pressure may be supposed to remain parallel to itself whilst that space is described.

† See Poncelet, *Mécanique Industrielle*, troisième partie.

of work it is capable of reproducing upon any resistance which may be opposed to its motion, and bring it to rest. A very simple investigation (Art. 66.) establishes the truth of this interpretation, and gives to the principle of *vis viva* the following new and more simple enunciation:—"The difference between the aggregate work done upon the machine, during any time, by those forces which tend to accelerate the motion, and the aggregate work, during the same time, of those which tend to retard the motion, is equal to the aggregate number of units of work accumulated in the moving parts of the machine during that time if the former aggregate exceed the latter, and lost from them during that time if the former aggregate fall short of the latter." Thus, then, if the aggregate work of the forces which tend to accelerate the motion of a machine exceeds that of the forces which tend to retard it, then is the surplus work (that done upon the driving points, above that expended upon the prejudicial resistances and upon the working points) continually accumulated in the moving elements of the machine, and their motion is thereby continually accelerated. And if the former aggregate be less than the latter, then is the deficiency supplied from the work already accumulated in the moving elements, so that their motion is in this case continually retarded.

The moving power divides itself whilst it operates in a machine, first, into that which overcomes the prejudicial resistances of the machine, or those which are opposed by friction and other causes, uselessly absorbing the work in its transmission. Se-

condly, into that which accelerates the motion of the various moving parts of the machine, and which accumulates in them so long as the work done by the moving power upon it exceeds that expended upon the various resistances opposed to the motion of the machine. Thirdly, into that which overcomes the useful resistances, or those which are opposed to the motion of the machine at the working point, or points, by the useful work which is done by it.

Between these three elements there obtains in every machine a mathematical relation, which I have called its MODULUS. The general form of this modulus I have discussed in a memoir on the "Theory of Machines" published in the *Philosophical Transactions* for the year 1841. The determination of the particular moduli of those elements of machinery which are most commonly in use is the subject of the third part of the following work. From a combination of the moduli of any such elements there results at once the modulus of the machine compounded of them.

When a machine has acquired a state of uniform motion work ceases to accumulate in its moving elements, and its modulus assumes the form of a direct relation between the work done by the motive power upon its driving point and that yielded at its working points. I have determined by a general method* the modulus in this case, from that statical relation between the driving and working pressures upon the machine which obtains in the state bordering

* Art. 152. See *Phil. Trans.*, 1841, p. 290.

upon its motion, and which may be deduced from the known conditions of equilibrium and the established laws of friction. In making this deduction I have, in every case, availed myself of the following principle, first published in my paper on the theory of the arch read before the Cambridge Philosophical Society in Dec. 1833, and printed in their *Transactions* of the following year:—"In the state bordering upon motion of one body upon the surface of another, the resultant pressure upon their common surface of contact is inclined to the normal, at an angle whose tangent is equal to the coefficient of friction."

This angle I have called the limiting angle of resistance. Its values calculated, in respect to a great variety of surfaces of contact, are given in a table at the conclusion of the second part, from the admirable experiments of M. Morin*, into the mechanical details of which precautions have been introduced hitherto unknown to experiments of this class, and which have given to our knowledge of the laws of friction a precision and a certainty hitherto unhoped for.

Of the various elements of machinery those which rotate about cylindrical axes are of the most frequent occurrence and the most useful application; I have, therefore, in the first place sought to establish the general relation of the state bordering upon motion between the driving and the working pressures upon such a machine, reference being had to the weight of

* *Nouvelles Expériences sur le Frottement*, Paris, 1833.

[redacted]

to wheels having epicycloidal and involute teeth, the modulus assumes a character of mathematical exactitude and precision, and at once establishes the conclusion (so often disputed) that the loss of power is greater before the teeth pass the line of centres than *at corresponding points* afterwards; that the contact should, nevertheless, in all cases take place partly before and partly after the line of centres has been passed. In the case of involute teeth, the proportion in which the arc of contact should thus be divided by the line of centres is determined by a simple formula; as also are the best dimensions of the base of the involute, with a view to the most perfect economy of power in the working of the wheels.

The greater portion of the subjects discussed in the third part of my work I believe to be entirely new to science. In the fourth part I have treated of "the theory of the stability of structures," referring its conditions, so far as they are dependent upon rotation, to the properties of a certain line which may be conceived to traverse every structure, passing through those points in it where its surfaces of contact are intersected by the resultant pressures upon them. To this line, whose properties I first discussed in a memoir upon "the Stability of a System of Bodies in Contact," printed in the sixth volume of the *Camb. Phil. Trans.*, I have given the name of

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direction of the mutual pressures of the teeth is determined by a method first applied by me to that purpose in a popular treatise, entitled *Mechanics applied to the Arts*, published in 1834.

to the determination of the pressure of earth upon revêtement walls, and a modification of that principle, suggested by M. Poncelet, to the determination of the resistance opposed to the overthrow of a wall backed by earth. This determination has an obvious application to the theory of foundations.

In the application of the principle of Coulomb I have availed myself, with great advantage, of the properties of the limiting angle of resistance. All my results have thus received a new and a simplified form.

The theory of the arch I have discussed upon principles first laid down in my memoir on "the Theory of the Stability of a System of Bodies in Contact," before referred to, and subsequently in a memoir printed in the "Treatise on Bridges" by Professor Hosking and Mr. Hann.* They differ essentially from those on which the theory of Coulomb is founded†; when, nevertheless, applied to the case treated by the French mathematicians they lead to identical results. I have inserted at the conclusion of my work the tables of the thrust of circular arches, calculated by M. Garidel from formulæ founded on the theory of Coulomb.

The fifth part of the work treats of the "strength

* I have made extensive use of the memoir above referred to in the following work, by the obliging permission of the publisher, Mr. Weale.

† The theory of Coulomb was unknown to me at the time of the publication of my memoirs printed in the *Camb. Phil. Trans.* For a comparison of the two methods see Mr. Hann's treatise.

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ERRATA.

Page 36. line 2. from bottom, *for* $x_1^{\frac{2}{3}}$ *read* $x_1^{\frac{3}{2}}$.
55. line 5. from bottom, *for* ABDC *read* AEFC.
64. line 3. from bottom, *for* Y_2 *read* y_2.
64. line 5. from bottom, *for* Y_{n-0} *read* Y_{n-1}.
122. line 10. from top, *for* half *read* double.
167. line 8. from top, *for* B_1 *read* B.
172. line 3. from top, *for* 114 *read* 119.
173. line 5. from bottom, *for* $\frac{P_2 a_2^2}{a_2^2} V_1^2$ *read* $\frac{P_2 a_2^2}{a_1^2} V_1^2$.
174. line 8. from top, *for* b^2 *read* b_2.
521. line 6. from top, *for* $\frac{1}{2}$ in. square *read* $\frac{3}{16}$ in. by $\frac{7}{16}$ in.

In the table page 152. the words "without unguent" enclosed by a bracket opposite to the words "iron upon oak," belong (with the corresponding numbers) to the following bracket.

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THE

MECHANICAL PRINCIPLES

OF

CIVIL ENGINEERING.

PART I.

STATICS.

1. FORCE is that which *tends* to cause or to destroy motion, or which actually causes or destroys it.

The *direction* of a force is that straight line in which it tends to cause motion in the point to which it is applied, or in which it tends to destroy the motion in it.

When more forces than one are applied to a body, and their respective tendencies to communicate motion to it counteract one another, so that the body remains at rest, these forces are said to be in EQUILIBRIUM, and are called PRESSURES.

It is found by experiment, that the effect of a pressure when applied to a solid body, is the same at whatever point in the line of its direction it is applied; so that the conditions of the equilibrium of that pressure, in respect to other pressures applied to the same body, are not altered, if, without altering the direction of the pressure, we remove its point of application, provided only the point to which we remove it be in the straight line in the direction of which it acts.

The science of STATICS is that which treats of the *equilibrium of pressures.* When two pressures *only* are applied to a body, and hold it at rest, it is found by experiment that

G be the centre of gravity of the whole trapezoid, and draw GM perpendicular to AD. Then would the whole be supported by a single force equal to the weight of the trapezoid acting upwards at G. Therefore (Art. 17.),

$$\overline{MG} \cdot \overline{ABCD} = \overline{G_1M_1} \cdot \overline{ABED} + \overline{G_2M_2} \cdot \overline{CED}$$

Now, $\quad ABCD = \frac{1}{2} h (a+b)$, $ABED = ha$,

$$CED = \tfrac{1}{2} h (b-a),\quad G_1M_1 = \tfrac{1}{2} h \quad G_2M_2 = \tfrac{2}{3} h,$$

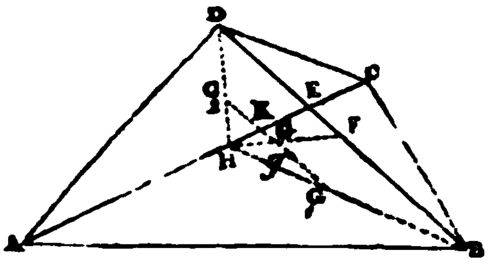
$$\therefore \overline{MG} \cdot \tfrac{1}{2} h (a+b) = \tfrac{1}{2} h \cdot ha + \tfrac{2}{3} h \cdot \tfrac{1}{2} h (b-a),$$

$$\therefore \overline{MG} (a+b) = ha + \tfrac{2}{3} h (b-a) = \tfrac{1}{3} h (a+2b),$$

$$\therefore MG = \tfrac{1}{3} h \cdot \frac{a+2b}{a+b} \quad \ldots\ldots \quad (19).$$

30. THE CENTRE OF GRAVITY OF ANY QUADRILATERAL FIGURE.

Draw the diagonals AC and BD of any quadrilateral figure ABCD, and let them intersect in E, and from the greater of the two parts, BE and DE, of either diagonal BD set off a part BF equal to the less part. Bisect the other diagonal AC in H, join HF and take HG equal to one third of HF; then will G be the centre of gravity of the whole figure.



For if not, let g be the centre of gravity, join HB and HD and take $HG_1 = \frac{1}{3}$ HB and $HG_2 = \frac{1}{3}$ HD, then will G_1 and G_2 be the centres of gravity of the triangles ABC and ADC respectively (Art. 25.). Suppose these triangles to be collected in their centres of gravity G_1, G_2; it is evident that the centre of gravity g, of the whole figure, will be in the straight line joining the points G_1, G_2: let this line intersect AC in K; then since a pressure equal to the weight of the whole figure acting upwards at g, will be in equilibrium with the weights of the triangles collected in G_1 and G_2, we have, by the principle of the equality of moments. (Art. 15.)

$$\overline{Kg} \cdot \overline{ABCD} = \overline{KG_1} \cdot \overline{ABC} - \overline{KG_2} \cdot \overline{ADC}.$$

Now since $HG_1 = \frac{1}{3}$ HB, and $HG_2 = \frac{1}{3}$ HD, therefore G_1G_2 is

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of the centre of gravity of the body from the given plane; since $\mu\Sigma x\Delta M$ represents the sum of the *moments* of a system of parallel pressures about that plane, μM the sum of those pressures, and G_1 the distance of their centre of pressure from the plane (Art. 19.), it follows by equation (18.) that

$$G_1 = \frac{\mu\Sigma x \, . \, \Delta M}{\mu M} = \frac{\Sigma x \, . \, \Delta M}{M} \quad \ldots\ldots \quad (20).$$

Now it is proved in the theory of the integral calculus*, that a sum, such as is represented by the above expression $\Sigma x\Delta M$, whose terms are infinite in number, and each the product of a finite quantity x, and an infinitely small quantity ΔM, and in which M is, as in this case, a function of x (and therefore x a function of M), is equal to the definite integral $\int_{x_2}^{x_1} x dM$. Therefore, generally,

Similarly,

$$\left.\begin{aligned} G_1 &= \frac{\int_{x_2}^{x_1} x dM}{M} \\ G_2 &= \frac{\int_{y_2}^{y_1} y dM}{M} \\ G_3 &= \frac{\int_{z_2}^{z_1} z dM}{M} \end{aligned}\right\} \quad \ldots\ldots \quad (21).$$

In the two last of which equations y and z are taken to represent, respectively, the distances of the element ΔM of the body from two other planes, as x represents its distance from

* Poisson, Journal de l'Ecole Polytechnique, 18me cahier, p. 320., or Art. 2. in the Treatise on Definite Integrals in the Encyclopædia Metropolitana by the author of this work. See Appendix, note A.

The centre of gravity of such an arc is evidently in the radius [illegible], which [illegible] since the arc is symmetrical about that radius. [illegible] a plane [illegible] perpendicular to this radius, and passing through the centre, to measure the moments from. [illegible] represent the distance [illegible] of any point [illegible] in the arc from this plane; and let [illegible] represent the arc [illegible] and [illegible] the arc [illegible], [illegible] the radius [illegible], and [illegible] the chord [illegible].

[illegible]

[illegible]

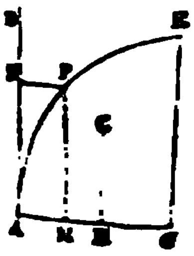
the integral being taken between the limits [illegible] and [illegible], because these are the values of [illegible] which correspond to the extreme points [illegible] and [illegible] of the arc.

Now, [illegible]

$$\therefore G = \frac{ac}{S} \quad \ldots\ldots \quad (25).$$

The distance of the centre of gravity of a circular arc from the centre of the circle is therefore a fourth proportional to the length of the arc, the length of the chord, and the radius of the arc.

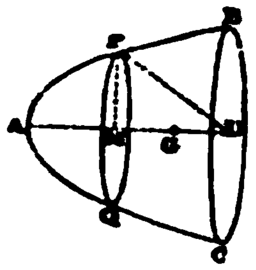
*33. THE CENTRE OF GRAVITY OF A CURVILINEAR AREA WHICH LIES WHOLLY IN THE SAME PLANE.



Let BAC represent such an area. [illegible] represent the perpendicular distances [illegible] of any point P in the curve AB from [illegible] AC and AD, perpendicular to the plane [illegible] given area and [illegible]

If, then, G be the centre of gravity of the parabolic area ACB, then $AH=\frac{3}{5}AC$, $HG=\frac{3}{8}CB$.

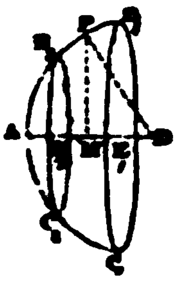
*34. The centre of gravity of a surface of revolution.



Any surface of revolution BAC is evidently symmetrical about its axis of revolution AD, its centre of gravity is therefore in that axis. Let the moments be measured from a plane passing through A and perpendicular to the axis AD, and let x and y be co-ordinates of any point P in the generating curve APB of the surface, and s the length of the curve AP. Then M being taken to represent the area of the surface, and being supposed to be made up of bands parallel to PQ, the area ΔM of each such band is represented (see p. 44.)* by $2\pi y\Delta s$, and its moment by $2\pi\mu xy\Delta s$,

$$\therefore G_1=\frac{2\pi\Sigma xy\Delta s}{M}=\frac{2\pi\int_{S_2}^{S_1}xyds}{M} \quad \ldots\ldots (26).$$

Example.—*To determine the centre of gravity of the surface of any zone or segment of a sphere.*



Let B_1AC_1 represent the surface of a sphere, whose centre is D, and whose radius DP is represented by a, and the arc AP by s. Then $x=DM=DP\cos. PDM=a\cos.\frac{s}{a}$, $y=PM=DP\sin. PDM=a\sin.\frac{s}{a}$, $\therefore 2xy=2a^2\sin.\frac{s}{a}\cos.\frac{s}{a}=a^2\sin.\frac{2s}{a}$.

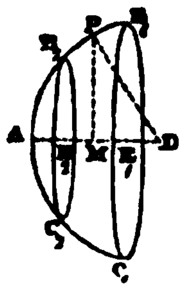
* Or Prof. Hall's Diff. Calculus, p. 168.

$$\therefore 2\pi \int_{S_2}^{S_1} xy\,ds = \pi a^2 \int_{S_2}^{S_1} \sin.\frac{2s}{a}\,ds$$

$$= \tfrac{1}{2}\pi a^3 \left\{ \cos.\frac{2S_2}{a} - \cos.\frac{2S_1}{a} \right\}$$

$$= \tfrac{1}{2}\pi a^3 \left\{ \left(1+\cos.\frac{2S_2}{a}\right) - \left(1+\cos.\frac{2S_1}{a}\right) \right\}$$

$$= \pi a^3 \left\{ \cos.^2\frac{S_2}{a} - \cos.^2\frac{S_1}{a} \right\} \quad \ldots\ldots (27).$$



where S_1 and S_2 are the values of s at the points B_1 and B_2, where the zone is supposed to terminate.

Also, since $\quad \dfrac{dM}{ds} = 2\pi y, \qquad \therefore M = 2\pi \displaystyle\int_{S_2}^{S_1} y\,ds$

$$= 2\pi a \int_{S^2}^{S^1} \sin.\frac{s}{a}\,ds = 2\pi a^2 \left\{ \cos.\frac{S_2}{a} - \cos.\frac{S_1}{a} \right\},$$

$$\therefore G_1 = \frac{1}{2}a \left\{ \cos.\frac{S_2}{a} + \cos.\frac{S_1}{a} \right\}$$

$$= \frac{1}{2}\left\{ DE_2 + DE_1 \right\} = DE \quad \ldots\ldots (28),$$

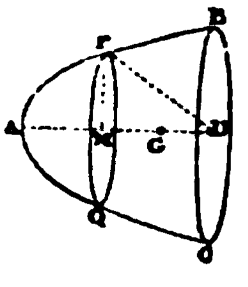
if E be the bisection of E_1E_2.

If $S_2=0$, or the zone commence from A, then

$$G_1 = \frac{1}{2}a\left\{1+\cos.\frac{S_1}{a}\right\} = a\cos.^2\frac{S_1}{2a} \quad \ldots\ldots (29).$$

*35. The centre of gravity of a solid of revolution.

Any solid of revolution BAC is evidently symmetrical

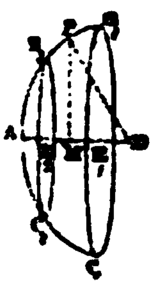


about its axis of revolution AD, its centre of gravity is therefore in that line; and taking a plane passing through A and perpendicular to that axis as the plane from which the moments are measured, we have only to determine the distance AG of the centre of gravity, from that plane.

Now, if x and y represent the co-ordinates of any point P in the generating curve, and M the volume of the portion PAQ of this solid, then, conceiving it to be made up of cylindrical laminæ parallel to PQ, the thickness of each of which is Δx, the volume of each is represented by $\pi y^2 \Delta x$, and its moment by $\pi \mu x y^2 \Delta x$.

$$\therefore \; G^1 = \frac{\pi \Sigma y^2 \Delta x}{M} = \frac{\pi \int_{x_2}^{x_1} x y^2 dx}{M} \; \dots\dots \; (30).$$

EXAMPLE.—*To determine the centre of gravity of any solid segment of a sphere.*



Let $B_1 A C_1$ represent any such segment of a sphere whose centre is D and its radius a. Let x and y represent the co-ordinates AM and MP of any point P, x being measured from A; then by the equation to the circle $y^2 = 2ax - x^2$,

$$\therefore \; \pi \int_{x_2}^{x_1} x y^2 dx = \pi \int_0^{x_1} x\,(2ax - x^2) dx = \pi(\tfrac{2}{3} a x_1^3 - \tfrac{1}{4} x_1^4).$$

$$\text{Also, } M = \pi \int_{x_2}^{x_1} y^2 dx = \pi \int_0^{x_1} (2ax - x^2) dx = \pi(a x_1^2 - \tfrac{1}{3} x_1^3),$$

$$\therefore \; G_1 = \frac{\tfrac{2}{3} a x_1 - \tfrac{1}{4} x^2}{a - \tfrac{1}{3} x_1} = \tfrac{1}{4} x_1 \, . \left(\frac{8a - 3x_1}{3a - x_1}\right) \; \dots\dots \; (31).$$

If the segment become a hemisphere, $x_1 = a$, $\therefore$ $G_1 = \frac{5}{8} a$.

which by this velocity of projection it can be made to describe.

If the body's motion be *accelerated*, and it fall from *rest*, or have no velocity of projection, then $v^2 - 0 = +2fS$,

$$\therefore \quad v^2 = 2fS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

Let S_2 be the space through which it must in this case move to acquire a velocity V equal to that with which it was projected in the last case, therefore $V^2 = 2fS_2$. Whence it follows that $S_1 = S_2$, or that the whole space S_1 through which a body will move when projected with a given velocity V, and uniformly *retarded* by any force, is equal to the space S_2, through which it must move to *acquire* that velocity when uniformly *accelerated* by the same force.

In the case of bodies moving freely, and acted upon by gravity, f equals $32\frac{1}{6}$ feet, and is represented by g; and the space S_2, through which any given velocity V is acquired, is then said to be that *due* to that velocity.

WORK.

48. Work is the union of a continued pressure with a continued motion. And a mechanical agent is thus said to work when a pressure is continually overcome, and a point (to which that pressure is applied) continually moved by it. Neither pressure nor motion alone is sufficient to constitute *work;* so that a man who merely supports a load upon his shoulders without moving it, no more *works*, in the sense in which that term is here used, than does a column which sustains a heavy weight upon its summit; and a stone as it falls freely *in vacuo*, no more *works* than do the planets as they wheel unresisted through space.

49. The unit of work. The unit of work used in this

If the body, instead of being accelerated, had been *retarded*, then the work lost being that expended in overcoming the retarding forces, is evidently that necessary to move the body uniformly in opposition to these retarding forces through AB; so that if this force be represented by U, then, since $\frac{1}{2}\frac{W}{g}(v^2-V^2)$ is in this case the work lost, we shall have $v^2-V^2=\frac{2gU}{W}$. Therefore, generally,

$$V^2-v^2=\pm\frac{2gU}{W} \quad \ldots\ldots \quad (46),$$

where the sign $\pm$ is to be taken according as the motion is accelerated or retarded.

69. *The work accumulated in a body which has moved through any space acted upon by any force, is equal to the excess of the work which has been done upon it by those forces which tend to accelerate its motion above that which has been done upon it by those which tend to retard its motion.*

For let R be the single force which would at any point P (see last fig.) be necessary to move the body back again through an exceeding small element of the same path (the other forces impressed upon it remaining as before); then it follows by Art. 54. that the work of R over this element of the path is equal to the excess of the work over that element of the forces which are impressed upon the body in the direction of its motion above the work of those impressed in the opposite direction. Now this is true at *every* point of the path; therefore the *whole* work of the force R necessary to move the body back again from B to A is equal to the excess of the work done upon it, by the impressed forces in the direction of its motion, above the work done upon it by them in a direction opposed to its motion; whence

$\therefore$ m^t in^a of prism about AB $=\frac{ab}{c}\int_{-\frac{1}{3}c}^{+\frac{2}{3}c}(\frac{2}{3}c-x)x^2dx+\frac{1}{12}\frac{ba^3}{c^3}\int_{-\frac{1}{3}c}^{+\frac{2}{3}c}(\frac{2}{3}c-x)^3dx.$

Performing the integrations here indicated, and representing the inertia of the prism about AB by I, we have

$$\mathrm{I}=\tfrac{1}{12}abc(\tfrac{1}{4}a^2+\tfrac{1}{3}c^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (63).$$

*85. *The moment of inertia of a solid cylinder about its axis of symmetry.*

Let AB be the axis of such a cylinder, whose radius AC is represented by a, and its height by b. Conceive the cylinder to be made up of cylindrical rings having the same axis; let AP$=\rho$ be the internal radius of one of these, and let its thickness PQ be represented by $\Delta\rho$, so that $\rho+\Delta\rho$ is the external radius AQ of the ring. Then will the *volume* of the ring be represented by $\pi b(\rho+\Delta\rho)^2-\pi b\rho^2$, or by $\pi b[2\rho\Delta\rho+(\Delta\rho)^2]$; or if $\Delta\rho$ be taken exceedingly small, so that $(\Delta\rho)^2$ may vanish as compared with $2\rho\Delta\rho$, then is the volume of the ring represented by $2\pi b\rho\Delta\rho$.

Now this being the case, the ring may be considered as an element ΔM of the volume of the solid, every part of which element is at the same distance ρ from the axis AB, so that the whole moment of inertia $\Sigma\rho^2\Delta\mathrm{M}$ of the cylinder $=\Sigma\rho^2(2\pi b\rho\Delta\rho)=2\pi b\Sigma\rho^3\Delta\rho$,

$$\therefore \quad \mathrm{I}=2\pi b\int_0^a\rho^3d\rho=\tfrac{1}{2}\pi ba^4 \quad . \quad . \quad . \quad . \quad . \quad (64).$$

*86. *The moment of inertia of a hollow cylinder about its axis of symmetry.*

Let a_1 be the external radius AC, and a_2 the internal

radius AP, and b the height of the cylinder; then by the last proposition the moment of inertia of the cylinder CD, if it were solid, would be $\frac{1}{2}\pi b a_1^4$; also the moment of inertia of the cylinder PR, which is taken from this solid to form the hollow cylinder, would be $\frac{1}{2}\pi b a_2^4$. Now let I represent the moment of inertia of the hollow cylinder CP, therefore $I+\frac{1}{2}\pi b a_2^4=\frac{1}{2}\pi b a_1^4$,

$$\therefore \; I=\tfrac{1}{2}\pi b(a_1^4-a_2^4)=\tfrac{1}{2}\pi b(a_1^2-a_2^2)(a_1^2+a_2^2)=\tfrac{1}{2}\pi b(a_1-a_2)(a_1+a_2)(a_1^2+a_2^2).$$

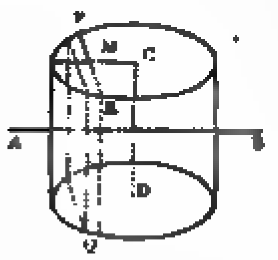
Let the thickness a_1-a_2 of the hollow cylinder be represented by c, and its mean radius $\frac{1}{2}(a_1+a_2)$ by R, therefore $a_1=R+\frac{1}{2}c$, $a_2=R-\frac{1}{2}c$.

Substituting these values in the preceding equation, we obtain

$$I=2\pi bcR\{R^2+\tfrac{1}{4}c^2\} \quad . \; . \; . \; . \; . \quad (65).$$

*87. *The moment of inertia of a cylinder about an axis passing through its centre of gravity, and perpendicular to its axis of symmetry.*

Let AB be such an axis, and let PQ represent a lamina contained between planes perpendicular to this axis, and exceedingly near to each other. Let CD, the axis of the cylinder, be represented by b, its radius by a, and let $CM=x$. Take Δx to represent the thickness of the lamina, and let $MP=y$. Now this lamina



may be considered a rectangular parallelopiped traversed through its centre of gravity by the axis AB; therefore by equation (62) its moment of inertia about that axis is represented by $\frac{1}{12}(\Delta x)b(2y)\{b^2+(2y)^2\}=\frac{1}{6}b\{b^2y+4y^3\}\Delta x$. Now the whole moment of inertia I of the cylinder about AB is evidently equal to the sum of the moments of inertia of all such laminæ;

$$\therefore \; I=\tfrac{1}{6}b\Sigma\{b^2y+4y^3\}\Delta x=\tfrac{1}{6}b\int_{-a}^{a}(b^2y+4y^3)dx.$$

96. The velocity V at any instant of a body moving with a *variable* motion, being the space which it would describe in a second of time, if at that instant its motion were to become uniform, it follows, that if we represent by Δt any number of seconds or parts of a second, beginning from that instant, and by ΔS, the space which the body would describe in the time Δt, if its motion continued uniform from the commencement of that time, then,

$$V\Delta t=\Delta S, \quad \therefore V=\frac{\Delta S}{\Delta t}.$$

Now this is true if the motion remain uniform during the time Δt, however small that time may be, and therefore if it be *infinitely* small. But if the time Δt be *infinitely* small, the motion does remain uniform during that time, however variable may be the moving force; also when Δt is infinitely small, $\frac{\Delta S}{\Delta t}=\frac{dS}{dt}$. Therefore, generally,

$$V=\frac{dS}{dt} \quad \ldots\ldots \quad (74).$$

The equations (73) and (74) are the fundamental equations of dynamics: they involve those dynamical results which have been discussed on other principles in the preceding parts of this work.*

The Descent of a Body upon a Curve.

*97. *If the moving force* P *upon a body varies directly as its*

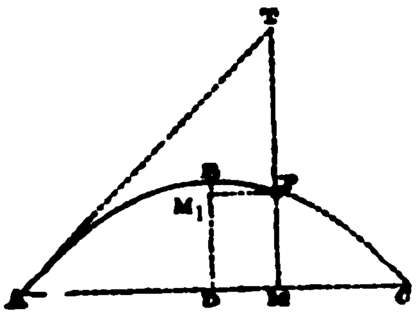
* Thus if the latter equation be inverted, and multiplied by the former, we obtain the equation

$$P\frac{dS}{dt}=\pm\frac{W}{g}\cdot V\left(\frac{dV}{dt}\right)=\pm\frac{W}{2g}\left(\frac{dV^2}{dt}\right),$$

$$\therefore \frac{dV^2}{dS}=\pm\frac{2g}{W}P,$$

$$\therefore V^2-v^2=\pm\frac{2g}{W}\int_{S_2}^{S_1}P\,dS,$$

which is identical with equation (47).



114. Let AM $=x$, MP $=y$, angle of projection TAM $=\alpha$, velocity of projection $=$ V.

$$\therefore\ x \sec.\alpha = \overline{AT} = V\ .\ T,\ \therefore\ T = \frac{x \sec.\alpha}{V},$$

$$x \tan.\alpha - y = \overline{MT} - \overline{MP} = \overline{TP} = \tfrac{1}{2}gT^2 \ .\ .\ .\ .\ .\ (91).$$

Substituting the value of T from the preceding equation,

$$x \tan.\alpha - y = \tfrac{1}{2}g\frac{x^2 \sec.^2\alpha}{V^2},$$

$$\therefore\ y = x \tan.\alpha - \frac{g \sec.^2\alpha}{2V^2}\ .\ x^2.$$

Let H be the height through which a body must fall freely by gravity to acquire the velocity V, or the height due to that velocity; then $V^2 = 2gH$ (Art. 47.), therefore $4H = \frac{2V^2}{g}$; therefore, by substitution,

$$y = x \tan.\alpha - \frac{\sec.^2\alpha}{4H}x^2\ .\ .\ .\ .\ .\ (92).$$

115. *To find the time of the flight of a projectile.*

It has been shown (equation 91), that if T represent the time in seconds of the flight to a point whose co-ordinates are x and y, then

$$\tfrac{1}{2}gT^2 = x \tan.\alpha - y,\quad \therefore\ T^2 = \frac{2}{g}\{x \tan.\alpha - y\},$$

$$\therefore\ T = \sqrt{\frac{2}{g}}\sqrt{x \tan.\alpha - y}\ .\ .\ .\ .\ .\ (93).$$

Now, $\frac{2}{g} = \frac{2}{32\frac{1}{6}} = \frac{1}{16}$ nearly, $\therefore\ T = \frac{1}{4}\sqrt{x \tan.\alpha - y}$ nearly.

If the projectile descend again to the horizontal plane from which it was projected, and T be the whole time of its flight,

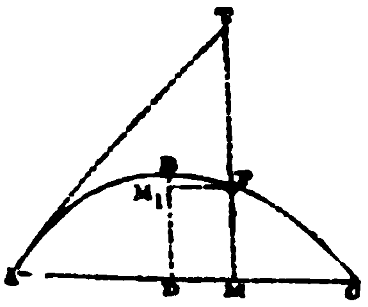
[redacted]

and X its whole range upon the plane, then, since at the expiration of the time T, $y=0$ and $x=X$,

$$\therefore \ T=\sqrt{\frac{2}{g}}\sqrt{X \tan.\alpha}=\tfrac{1}{4}\sqrt{X \tan.\alpha} \text{ nearly.}$$

116. *To find the greatest horizontal distance* X, *to which a projectile ranges, having given the elevation* α *and the velocity* V *of its projection.*

When the projectile attains its greatest horizontal range, its height y above the horizontal plane becomes 0, whilst the abscissa x of the point P, which it has then reached in its path, becomes X. Substituting these values 0 and X, for y and x in equation (92), we have $0=X \tan.\alpha-\frac{X^2 \sec.^2\alpha}{4H}$,

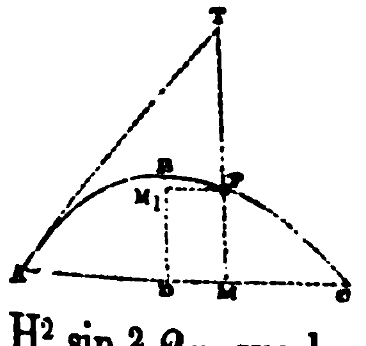


$$\therefore X=4H \tan.\alpha \cos.^2\alpha=4H \sin.\alpha \cos.\alpha.$$

$$\therefore X=2H \sin. 2\alpha \quad . \ . \ . \ . \quad (94).$$

If the body be projected at different angular elevations, but with the same velocity, the horizontal range will be the greatest when $\sin. 2\alpha$ is the greatest, or when $2\alpha=\frac{\pi}{2}$, or $\alpha=\frac{\pi}{4}$.

117. *To find the greatest height which a projectile will attain in its flight if projected with a given velocity, and at a given inclination to the horizon.*

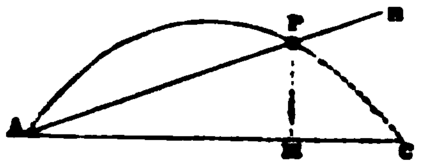


Multiplying both sides of equation (92) by $4H \cos.^2\alpha$, we have $4H \cos.^2\alpha \,.\, y=4H \cos.^2\alpha \tan.\alpha \,.\, x-x^2=2H\,(2 \cos.\alpha \sin.\alpha)\,x-x^2=2H \sin. 2\alpha \,.\, x-x^2$. Subtracting both sides of this equation from $H^2 \sin.^2 2\alpha$, we have

$$H^2 \sin.^2 2\alpha-4H \cos.^2\alpha \,.\, y=H^2 \sin.^2 2\alpha-2H \sin. 2\alpha \,.\, x+x^2.$$

119. *To find the range of a projectile upon an inclined plane.*

Let R represent the range AP of a projectile upon an inclined plane AB, whose inclination is ι. Then H and α being taken to represent the same quantities as before, and x, y being the co-ordinates of P to the horizontal axis AC, we have



$$x = \text{AM} = \text{AP cos. PAM} = \text{R cos. } \iota,$$
$$y = \text{PM} = \text{AP sin. PAM} = \text{R sin. } \iota.$$

Substituting these values of x and y in the general equation (92) to the projectile, we have

$$\text{R sin. } \iota = \text{R cos. } \iota \text{ tan. } \alpha - \frac{\text{R}^2 \text{ cos.}^2 \iota \text{ sec.}^2 \alpha}{4\text{H}}.$$

Dividing by R, multiplying by cos. α, and transposing

$$\frac{\text{R cos.}^2 \iota \text{ sec. } \alpha}{4\text{H}} = \text{cos. } \iota \text{ sin. } \alpha - \text{sin. } \iota \text{ cos. } \alpha = \text{sin. } (\alpha - \iota),$$

$$\therefore \text{ R} = 4\text{H} \frac{\text{sin. } (\alpha - \iota) \text{ cos. } \alpha}{\text{cos.}^2 \iota} \quad \ldots\ldots (97).$$

Now $\text{sin. } (2\alpha - \iota) - \text{sin. } \iota = \text{sin. } \{\alpha + (\alpha - \iota)\} - \text{sin. } \{\alpha - (\alpha - \iota)\} = 2 \text{ sin. } (\alpha - \iota) \text{ cos. } \alpha$.

Substituting this value of $2 \text{ sin. } (\alpha - \iota) \text{ cos. } \alpha$ in the preceding equation, we have

$$\text{R} = 2\text{H} \left\{ \frac{\text{sin. } (2\alpha - \iota) - \text{sin. } \iota}{\text{cos.}^2 \iota} \right\} \quad \ldots\ldots (98).$$

Now it is evident that if α be made to vary, ι remaining the same, R will attain its greatest value when sin. $(2\alpha - \iota)$ is greatest, that is when it equals unity, or when $2\alpha - \iota = \frac{\pi}{2}$, or when $\alpha = \frac{\pi}{4} + \frac{\iota}{2}$. This, then, is the angle of elevation corresponding to the greatest range, with a given velocity upon an inclined plane whose inclination is ι.

* The surfaces wear when there is no grease.

† The surfaces still retaining a little unctuousness.

‡ Ibid.

§ When the grease is constantly renewed and uniformly distributed, this proportion can be reduced to 0·05.

Surfaces in Contact.	State of the Surfaces.	Co-efficient of Friction when the Grease is renewed.		Limiting Angle of Resistance.
		In the usual Way.	Continuously.	
Lignum vitæ axles, ditto	coated with hog's lard	0·12	- -	6° 51′
	greasy	0·15	- -	8 32
Lignum vitæ axles in lignum vitæ bearings	coated with hog's lard	- -	0·07	4 0

Table IV.

Co-efficients of friction under pressures increased continually up to the limits of abrasion. From the experiments of Mr. G. Rennie.*

Pressure per Square Inch.	Co-efficients of Friction.			
	Wrought-iron upon Wrought-iron.	Wrought-iron upon Cast-iron.	Steel upon Cast-iron.	Brass upon Cast-iron.
32·5 lb.	·140	·174	·166	·157
1·66 cwt.	·250	·275	·300	·225
2·00	·271	·292	·333	·219
2·33	·285	·321	·340	·214
2·66	·297	·329	·344	·211
3·00	·312	·333	·347	·215
3·33	·350	·351	·351	·206
3·66	·376	·353	·353	·205
4·00	·376	·365	·354	·208
4·33	·395	·366	·356	·221
4·66	·403	·366	·357	·223
5·00	·409	·367	·358	·233
5·33		·367	·359	·234
5·66		·367	·367	·235
6·00		·376	·403	·233
6·33		·434		·234
6·66				·235
7·00				·232
7·33				·273

* Phil. Trans. 1829, table 8. p. 159.

TABLE V. RIGIDITY OF ROPES.

Table of the values of the constants D *and* E, *according to the experiments of Coulomb (reduced to English measures). The radius* R *of the pulley is to be taken in feet.*

No. 1. New dry cords. Rigidity proportional to the square of the circumference.

Circumference of the Rope in Inches.	Value of D in lbs.	Value of E in lbs.
1	·131528	[illegible]
2	·526108	·023030
4	2·104451	·073175
8	8·413702	·368494

Squares of proportions of the intermediate circumferences to those of the table.

No. 2. New ropes dipped in water. Rigidity proportional to the square of the circumference.

Circumference of the Rope in Inches.	Value of D in lbs.	Value of E in lbs.
1	·263053	·0057576
2	1·052217	[illegible]
[illegible]	4·208902	·0731755
8	16·835606	·3684660

No. 3. Dry half-worn ropes. Rigidity proportional to the square root of the cube of the circumference.

Circumference of the Rope in Inches.	Value of D in lbs.	Value of E in lbs.
1	·146272	·0064033
2	·413656	·0180827
4	1·169641	·0512115
8	3·308787	·1448238

Square roots of the cubes of proportions of the intermediate circumferences to those of the table.

No. 4. Wetted half-worn cords. Rigidity proportional to the square root of the cube of the circumference.

Circumference of the Rope in Inches.	Value of D in lbs.	Value of E in lbs.
1	·292541	·006401
2	·827328	·018107
4	2·339675	·051212
8	6·616589	·144822

Table VI.

Tarred rope. Rigidity proportional to the number of strands.

Number of Strands.	Value of D in lbs.	Value of E in lbs.
6	0·33390	0·009305
15	0·17212	0·021713
30	1·25294	0·044983

To determine the constants D and E for ropes whose circumferences are intermediate to those of the tables, find the ratio of the given circumference to that *nearest* to it in the tables, and seek this ratio or proportion in the first column of the auxiliary table to the right of the page. The corresponding number in the second column of this auxiliary table is a factor by which the values of D and E for the nearest circumference in the principal tables being multiplied, their values for the given circumference will be determined.

155. *The modulus of uniform motion in the wheel and axle.*

It is evident from equation (122), that, in the case of the wheel and axle, the relation assumed in equation (114) obtains, if we take $\Phi_1=\left(1+\frac{E}{a_2}\right)\frac{a_2+\rho\sin.\varphi}{a_1-\rho\sin.\varphi}$;

and
$$\Phi_2=\frac{D+\left(\frac{D}{a_2}\pm W\right)\rho\sin.\varphi}{a_1-\rho\sin.\varphi},$$

Now observing that $\Phi_1^{(0)}$ represents the value of Φ_1 when the prejudicial resistances vanish (or when $\varphi=0$ and $E=0$), we have $\Phi_1^{(0)}=\frac{a_2}{a_1}$.

$$\therefore\ \frac{\Phi_1}{\Phi^{(0)}}=\left(1+\frac{E}{a_2}\right)\frac{a_1}{a_2},\ \frac{a_2+\rho\sin.\varphi}{a_1-\rho\sin.\varphi}=\left(1+\frac{E}{a_2}\right)\frac{1+\left(\frac{\rho}{a_2}\right)\sin.\varphi}{1-\left(\frac{\rho}{a_1}\right)\sin.\varphi};$$

Therefore by equation (121),

$$U_1=U_2\left(1+\frac{E}{a_2}\right)\left\{\frac{1+\left(\frac{\rho}{a_2}\right)\sin.\varphi}{1-\left(\frac{\rho}{a_1}\right)\sin.\varphi}\right\}+S_1\left\{\frac{D+\left(\frac{D}{a_2}\pm W\right)\rho\sin.\varphi}{a_1-\rho\sin.\varphi}\right\}\ldots(12$$

which is the modulus of the wheel and axle.

Omitting terms involving dimensions of $\frac{\rho}{a_1}\sin.\varphi$, and $\frac{\rho}{a_2}\sin.\varphi$, and $\frac{E}{a_1}$ above the first, we have

$$U_1=U_2\left\{1+\frac{E}{a_2}+\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\rho\sin.\varphi\right\}+\frac{S_1D}{a_1}\left\{1+\left(\frac{1}{a_1}+\frac{1}{a_2}\pm\frac{W}{D}\right)\rho\sin.\varphi\right\}\ldots(125).$$

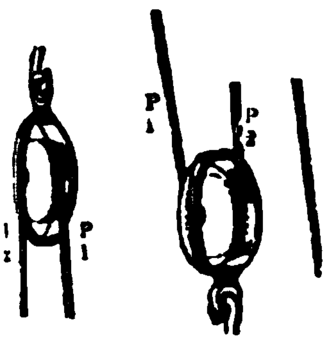
156. *The modulus of variable motion in the wheel and axle.*

If the relation of P_1 and P_2 be not that of either state bordering upon motion, then the motion will be continually accelerated or continually retarded, and work will continually accumulate in the moving parts of the machine, or the work

[redacted]

The Pulley.

158. If the radius of the axle be taken equal to that of the wheel, the wheel and axle becomes a pulley.



Assuming then in equation 122, $a_1 = a_2 = a$, we obtain for the relation of the moving pressures P_1 and P_2, in the state bordering upon motion in the pulley, when the strings are parallel,

$$P_1 = P_2\left(1+\frac{E}{a}\right)\left\{\frac{1+\frac{\rho}{a}\sin.\varphi}{1-\frac{\rho}{a}\sin.\varphi}\right\} + \frac{D+\left(\frac{D}{a}\pm W\right)\rho\sin.\varphi}{a-\rho\sin.\varphi} \quad . . (129);$$

and by equation 124 for the value of the modulus,

$$U_1 = U_2\left(1+\frac{E}{a}\right)\left\{\frac{1+\frac{\rho}{a}\sin.\varphi}{1-\frac{\rho}{a}\sin.\varphi}\right\} + S_1\left\{\frac{D+\left(\frac{D}{a}\pm W\right)\rho\sin.\varphi}{a-\rho\sin.\varphi}\right\} \quad . . (130);$$

in which the sign $\pm$ is to be taken according as the pressures P_1 and P_2 act downwards, as in the first pulley of the preceding figure; or upwards, as in the second. Omitting dimensions of $\frac{\rho}{a_1}\sin.\varphi$, $\frac{\rho}{a_2}\sin.\varphi$, and $\frac{E}{a}$ above the first, we have by equations (123, 125)

$$P_1 = P_2\left\{1+E+\frac{2\rho\sin.\varphi}{a}\right\} + \frac{D}{a}\left\{1+\left(\frac{2}{a}\pm\frac{W}{D}\right)\rho\sin.\varphi\right\} \quad . . (131),$$

$$U_1 = U_2\left\{1+E+\frac{2\rho\sin.\varphi}{a}\right\} + \frac{S_1 D}{a}\left\{1+\left(\frac{2}{a}\pm\frac{W}{D}\right)\rho\sin.\varphi\right\}. \quad (132).$$

Also observing that $a_1 = a_2$, and $I_2 = 0$, the modulus of variable motion (equation 126) becomes

$$U^1 = AU_2 + BS + \frac{1}{2g}(V_2^2 - V_1^2)\{P_1 + P_2 + \tfrac{1}{2}W\} \quad . . . . (133),$$

and the velocity of variable motion (equation 118, 127) is determined by the equation

Adding aT to both sides of the second of the above equations, and multiplying both sides by a, we have

$$a(1+a)T=a^2(T+t)+ab_1=a^2(P_2+W)+ab_1.$$

Also multiplying the first equation by $(1+a)$,

$$(1+a)P_1=a(1+a)T+b(1+a)=a^2(P_2+W)+ab_1+b(1+a),$$

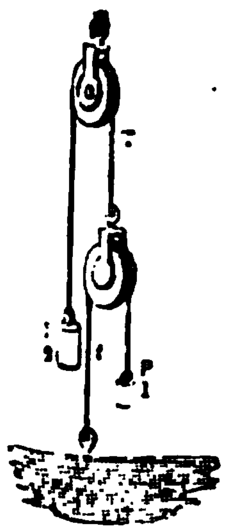
$$\therefore P_1=\left(\frac{a_2}{1+a}\right)P_2+\frac{a^2W+b(1+a)+ab_1}{1+a} \quad . \; . \; . \; . \; . \; (135).$$

Now if there were no friction or rigidity, a would evidently become 1 (see equation 131), and $\frac{a^2}{1+a}$ would become $\frac{1}{2}$; the co-efficients of the modulus (Art. 152.) are therefore

$$A=2\left(\frac{a^2}{1+a}\right), \text{ and } B=\frac{a^2W+b(1+a)+ab_1}{1+a};$$

$$\therefore U_1=2\left(\frac{a^2}{1+a}\right)U_2+\frac{a^2W+b(1+a)+ab_1}{1+a}S_1 \quad . \; . \; . \; . \; (136),$$

which is the modulus of uniform motion to the single moveable pulley.*



If this system of two pulleys had been arranged *thus*, with a different string passing over each, instead of with a single string as shown in the preceding figure, then, representing by t the tension upon the second part of the string to which P_1 is attached, and by T that upon the first part of the string to which P_2 is attached, we have

$$P_1=at+b, \qquad T=aP_2+b, \qquad P_1+t+W=T.$$

Multiplying the last of these equations by a, and adding it to the first, we have $P_1(1+a)+Wa=Ta+b=a\,P_2+(1+a)b$;

* The modulus may be determined directly from equation (135); for it is evident that if S_1 and S_2 represent the spaces described in the same time by P_1 and P_2, then $S_1=2S_2$. Multiplying both sides of equation (135) by this equation, we have,

$$P_1S_1=2\left(\frac{a^2}{1+a}\right)P_2S_2+\frac{a^2W+b(1+a)+ab_1}{1+a}\,2S_2;$$

now $P_1S_1=U_1$ and $P_2S_2=U_2$, therefore, &c.

$$\therefore\ P_1=\left(\frac{a^2}{1+a}\right)P_2+b-\frac{Wa}{1+a}\ .\ .\ .\ .\ .\ (137),$$

and for the modulus (equation 121),

$$U_1=2\left(\frac{a}{1+a}\right)U_2+\left(b-\frac{Wa}{1+a}\right)S_1\ .\ .\ .\ .\ .\ (138).$$

It is evident that, since the co-efficient of the second term of the modulus of this system is less than that of the first system (equation 136) (the quantities a and b being essentially positive), a given amount of work U_2 may be done by a less expense of power U_1, or a given weight P_2 may be raised to a given height with less *work*, by means of this system than the other; an advantage which is *not* due entirely to the circumstance that the weight of the moveable pulley in this case acts in *favour* of the power, whereas in the other it acts *against* it; and which advantage would exist, in a less degree, were the pulleys without weight.

A System of one fixed and any Number of moveable Pulleys.

160. Let there be a system of n moveable pulleys and one fixed pulley combined as shown in the figure, a separate string passing over each moveable pulley; and let the tensions on the two parts of the string which passes over the first moveable pulley be represented by T_1 and t_1, those upon the two parts of the string which passes over the second by T_2 and t_2, &c. Also, to simplify the calculation, let all the pulleys be supposed of equal dimensions and weights, and the cords of equal rigidity;

$$\therefore\ T_1=at_1+b_1,\ \text{and}\ T_2+W=T_1+t_1;$$

$$\therefore\ \text{eliminating,}\ T_1=\left(\frac{a}{1+a}\right)T_2+\frac{Wa+b_1}{1+a}\ .\ .\ .\ .\ .\ (139).$$

Let the co-efficients of this equation be represented by α and β;

$$\therefore\ T_1 = \alpha T_2 + \beta.$$

Similarly, $T_2 = \alpha T_3 + \beta$, $T_3 = \alpha T_4 + \beta$, $T_4 = \alpha T_5 + \beta$, &c. = &c., $T_{n-1} = \alpha T_n + \beta$, $T_n = \alpha P_2 + \beta$.

Multiplying these equations successively, beginning from the second, by α, α^2, α^3, &c., α^{n-1}, adding them together, and striking out terms common to both sides of the resulting equation, we have

$$T_1 = \alpha^n P_2 + \beta + \alpha\beta + \alpha^2\beta + \ldots\ldots + \alpha^{n-1}\beta\,;$$

or summing the geometrical progression in the second member,

$$T_1 = \alpha^n P_2 + \beta\left(\frac{\alpha^n - 1}{\alpha - 1}\right) \quad \ldots\ldots \quad (140)\,;$$

Substituting for α and β their values from equation (139), and reducing,

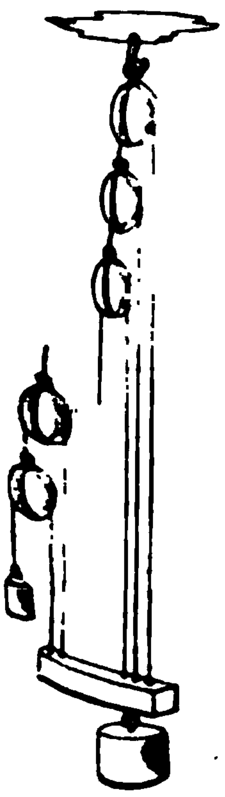
$$T_1 = \left(\frac{a}{1+a}\right)^n P_2 + (Wa + b_1)\left\{1 - \left(\frac{a}{1+a}\right)^n\right\}.$$

Now $P_1 = aT_1 + b$;

$$\therefore P_1 = a\left(\frac{a}{1+a}\right)^n P_2 + a(Wa + b_1)\left\{1 - \left(\frac{a}{1+a}\right)^n\right\} + b \ldots (141).$$

Whence observing, that, were there no friction, a would become unity, and $\left(\frac{a}{1+a}\right)^n = \left(\frac{1}{2}\right)^n$. We have (equation 121) for the modulus of this system,

$$U_1 = a\left(\frac{2a}{1+a}\right)^n U_2 + \left\{a(Wa + b_1)\{1 - \left(\frac{a}{1+a}\right)^n\} + b\right\} S_1 \ldots (142).$$



161. If each cord, instead of having one of its extremities attached to a fixed obstacle, had been connected by one extremity to a moveable bar carrying the weight P_2 to be raised (an arrangement which is shown in the second figure), then, adopting the same notation as before, we have

$$T_1 = at_1 + b, \qquad at_2 + b = T_2, \qquad T_2 = T_1 + t_1 + W.$$

Adding these equations together, striking out terms common to both sides, and solving in respect to T_1, we have

$$t_1=\left(\frac{a}{a+1}\right)t_2-\left(\frac{1}{a+1}\right)W;$$

in which equation it is to be observed, that the symbol b does not appear; that element of the resistance, (which is constant,) affecting the tensions t_1 and t_2 equally, and therefore eliminating with T_1 and T_2. Let $\frac{a}{a+1}$ be represented by α, then

$$\left.\begin{array}{l} t_1=\alpha t_2-\frac{\alpha}{a}W. \text{ Similarly, } t_2=\alpha t_3-\frac{\alpha}{a}W, \\ t_3=\alpha t_4-\frac{\alpha}{a}W, \&c.=\&c., \ t_{n-1}=\alpha t_n-\frac{\alpha}{a}W \end{array}\right\} \quad \ldots\ldots (143).$$

Eliminating between these equations precisely as between the similar equations in the preceding case (equation 140), observing only that here β is represented by $-\alpha W$, and that the equations (143) are $n-1$ in number instead of n, we have

$$t_1=\alpha^{n-1}t_n-\frac{\alpha W}{a}\left(\frac{\alpha^{n-1}-1}{\alpha-1}\right) \quad \ldots\ldots (144).$$

Also adding the preceding equations (143) together, we have

$$t_1+t_2+\ \ldots\ +t_{n-1}=\alpha(t_2+t_3+\ \ldots\ t_n)-(n-1)\frac{\alpha W}{a}.$$

Now the pressure P_2 is sustained by the tensions t_1, t_2, &c. of the different strings attached to the bar which carries it. Including in P_2, therefore, the weight of the bar, we have

$t_1+t_2+\ \ldots\ +t_{n-1}+t_n=P_2$; $\therefore\ t_1+t_2+\ \ldots\ +t_{n-1}=P_2-t_n$; and $t_2+\ \ldots\ +t_n=P_2-t_1$;

$$\therefore\ P_2-t_n=\alpha(P_2-t_1)-(n-1)\frac{\alpha W}{a}.$$

$$\therefore\ t_n=(1-\alpha)P_2+\alpha t_1+(n-1)\frac{\alpha W}{a}.$$

Substituting this value of t_n in equation (144),

$$t_1=(1-\alpha)\alpha^{n-1}P_2+\alpha^n t_1+(n-1)\frac{\alpha^n W}{a}-\frac{\alpha W}{a}\,\frac{\alpha^{n-1}-1}{\alpha-1}.$$

Transposing and reducing,

$$(1-\alpha^n)t_1=(1-\alpha)\alpha^{n-1}P_2+\frac{W}{a}\left\{n\alpha^n-\alpha\frac{1-\alpha^n}{1-\alpha}\right\};$$

$$\therefore\ t_1=\frac{(1-\alpha)\alpha^{n-1}}{1-\alpha^n}P_2+\frac{W}{a}\left\{\frac{n\alpha^n}{1-\alpha^n}-\frac{\alpha}{1-\alpha}\right\}.$$

Now $\alpha=\frac{a}{a+1}$, $\therefore\ \alpha^{-1}=1+a^{-1}$; also $\frac{(1-\alpha)\alpha^{n-1}}{1-\alpha^n}=\frac{\alpha^{-1}-1}{\alpha^{-n}-1}=$ $\frac{a^{-1}}{(1+a^{-1})^n-1}$, $\frac{\alpha^n}{1-\alpha^n}=\frac{1}{\alpha^{-n}-1}=\frac{1}{(1+a^{-1})^n-1}$, and $\frac{\alpha}{1-\alpha}=a$;

$$\therefore\ t_1=\frac{a^{-1}P_2}{(1+a^{-1})^n-1}+\frac{W}{a}\left\{\frac{n}{(1+a^{-1})^n-1}-a\right\}.$$

Now $P_1=at_1+b$;

$$\therefore\ P_1=\frac{P_2}{(1+a^{-1})^n-1}+\frac{W}{a}\left\{\frac{n}{(1+a^{-1})^n-1}-a\right\}+b\ldots(145).$$

Whence observing that when $a=1$, $\{(1+a^{-1})^n-1=2^n-1$, we obtain for the modulus of uniform motion (equation 121),

$$1=\left\{\frac{2^n-1}{(1+a^{-1})^n-1}\right\}.U_2+\left\{\left(\frac{W}{a}\right)\left\{\frac{n}{(1+a^{-1})^n-1}-a\right\}+b\right\}S_1\ldots(146).$$

A Tackle of any Number of Sheaves.

162. If any number of pulleys (called in this case sheaves) be made to turn on as many different centres in the same block A,

and if in another block B there be similarly placed as many others, the diameter of each of the last being one half that of a corresponding pulley or sheave in the first; and if the same cord attached to the first block be made to pass in succession over all the sheaves in the two blocks, as shown in the figure, it is evident that the parts of this cord 1, 2, 3, &c. passing between the two blocks, and as many in number as there are sheaves, will be parallel to each other, and will divide between them the pressure of a weight P_2 suspended from the lower block: moreover, that they would divide this pressure between them *equally* were it not

of the corresponding sheaf in the other *obliquely,* so that the parts of the cords between the blocks are not truly parallel to one another, and the sum of their tensions is not truly equal to the weight P_2 to be raised, but somewhat greater than it. So long, however, as the blocks are not very near to one another, this deflection of the cord is inconsiderable, and the error resulting from it in the calculation may be neglected. Supposing the different parts of the cord between the blocks then to be parallel, and the diameters of all the sheaves and their axes to be equal, also neglecting the influence of the *weight* of each sheaf in increasing the friction of its axis, since these weights are in this case comparatively small, the co-efficients a_1, a_2, a_3 will manifestly all be equal; as also b_1, b_2, b_3;

$$\therefore \left.\begin{array}{l} P_1=aT_1+b,\ T_1=aT_2+b,\ T_2=aT_3+b, \\ \&c.=\&c.,\ T_{n-1}=aT_n+b \end{array}\right\} \ldots\ldots (147);$$

also
$$P_2=T_1+T_2+T_3+\ldots\ldots+T_n.$$

Multiplying equations (147) successively (beginning from the second) by a, a^2, a^3, and a^{n-1}; then adding them together, striking out the terms common to both sides, and summing the geometric series in the second member (as in equation 140), we have

$$P_1=a^nT_n+b\frac{a^n-1}{a-1}.$$

Adding equations (147), and observing that $T_1+T_2+\ldots+T_n=P_2$, and that $P_1+T_1+T_2+\ldots+T_{n-1}=P_1+P_2-T_n$, we have

$$P_1+P_2-T_n=aP_2+nb.$$

Eliminating T_n between this equation and the last,

$$P_1=a^n\{P_1-P_2(a-1)-nb\}+b\frac{a^n-1}{a-1};$$

$$\therefore P_1=\frac{a^n(a-1)}{a^n-1}P_2+\frac{nba^n}{a^n-1}-\frac{b}{a-1} \ldots\ldots (148).$$

To determine the modulus let it be observed, that, neglecting

friction and rigidity, a becomes unity; and that for this value of a, $\frac{a^n(a-1)}{a^n-1}$ becomes a vanishing fraction, whose value is determined by a well known method to be $\frac{1}{n}$*. Hence (Art. 152.),

$$U_1 = n\frac{a^n(a-1)}{a^n-1}U_2 + \left\{\frac{nba^n}{a^n-1} - \frac{b}{a-1}\right\}S_1 \ldots\ldots (149).$$

Hitherto no account has been taken of the work expended in raising the rope which ascends with the ascending weight. The correction is, however, readily made. By Art. 60. it appears that the work expended in raising this rope (different parts of which are raised different heights) is precisely the same as though the whole quantity thus raised had been raised at one lift through a height equal to that through which its centre of gravity is actually raised. Now the cord raised is that which may be conceived to lie between two positions of P_2 distant from one another by the space S_2, so that its whole length is represented by nS_2; and if μ represent the weight of each foot of it, its whole weight is represented by $\mu n S_2$: also its centre of gravity is evidently raised between the first and second positions of P_2 by the distance $\frac{1}{2}S_2$; so that the whole work expended in raising it is represented by $\frac{1}{2}\mu n S_2^2$ or by $\frac{1}{2}\frac{\mu S_1^2}{n}$, since $S_1 = nS_2$. Adding this work expended in raising the rope to that which would be

* Dividing numerator and denominator of the fraction by $(a-1)$ it becomes $\frac{a^n}{a^{n-1}+a^{n-2}+\ldots+1}$, which evidently equals $\frac{1}{n}$ when $n=1$. The modulus may readily be determined from equation (148). Let S_1 and S_2 represent the spaces described by P_1 and P_2 in any the same time; then, since when the blocks are made to approach one another by the distance S_2, each of the n portions of the cord intercepted between the two blocks is shortened by this distance S_2, it is evident that the whole length of cord intercepted between the two blocks is shortened by nS_2; but the whole of this cord must have passed over the first sheaf, therefore $S_1 = nS_2$. Multiplying equation (148) by this equation, and observing that $U_1 = P_1S^1$ and $U_2 = P_2S_2$, we obtain the modulus as given above.

radius CK at the point K, where it intersects the axis at an angle CKR, equal to the limiting angle of resistance (see Art. 153.). Now, the resistance of the axis acts evidently in both cases in a direction opposite to the resultant of P_1 and P_2, and is equal to it; let it be represented by R. Upon the directions of P_1, P_2, and R, let fall the perpendiculars CA_1, CA_2, and CL, and let them be represented by a_1, a_2, and λ. Then, by the principle of the equality of moments, since P_1, P_2, and R are pressures in equilibrium,

$$\therefore P_1a_1 = P_2a_2 + \lambda R.$$

If P_1 had been upon the point of *yielding*, or P_2 on the point of preponderating, then R would have had its direction (in both cases) on the other side of C; so that the last equation would have become

$$P_1a_1 + \lambda R = P_2a_2.$$

According, therefore, as P_1 is in the superior or inferior state bordering upon motion,

$$P_1a_1 - P_2a_2 = (\pm\lambda)R.$$

And if we assume λ to be taken with the sign + or −, according as P_1 is about to preponderate or to yield, then *generally*

$$P_1a_1 - P_2a_2 = \lambda R \quad . \quad . \quad . \quad . \quad . \quad (155).$$

Now, since the resistance of the axis is equal to the resultant of P_1 and P_2, if we represent the angle P_1IP_2 by ι*, we have (Art. 13.)

$$R = \sqrt{P_1^2 + 2P_1P_2\cos.\iota + P_2^2}.$$

Substituting this value of R in the preceding equation, and squaring both sides,

$$(P_1a_1 - P_2a_2)^2 = \lambda^2(P_1^2 + 2P_1P_2\cos.\iota + P_2^2);$$

* Care must be taken to measure this angle, so that P_1 and P_2 may may have their directions both *towards* or both *from* the angular point I (as shown in the figure), and not one of them *towards* that point and the other *from* it. Thus, in the second figure, the inclination ι of the pressures P_1 and P_2 is not the angle A_2IP_1, but the angle P_2IP_1. It is of importance to observe this distinction (see note p. 190.).

It is evident that as the point A_1 thus continually alters its position, the distance A_1A_2 or L will continually change, so that the value of P_1 (equation 158.) will continually change. Now the work done under this variable pressure during one revolution of P_1 is represented (Art. 51.) by the formula $U_1=\int_0^{2\pi} P_1 a_1 d\theta$, if θ represent the angle A_1CA described at any time about C, by the perpendicular C_1A_1, and therefore $a_1\theta$, the space S described in the same time by the point of application A_1 of P_1 (see Art. 62.).

Substituting, therefore, for P_1 its value from equation (158.), we have

$$U_1=\int_0^{2\pi} P_2\left\{\left(\frac{a_2}{a_1}\right)+\left(\frac{\rho L}{a_1^2}\right)\sin.\varphi\right\}a_1 d\theta=$$

$$\int_0^{2\pi} P_2 a_2 d\theta+\frac{\rho\sin.\varphi}{a_1}\int_0^{2\pi} P_2\,.\,L d\theta;$$

$$\therefore\ U_1=U_2+\frac{\rho\sin.\phi}{a_1}\int_0^{2\pi} P_2\,.\,L d\theta\ .\ .\ .\ .\ .\ (192.)$$

Let now P_2 be assumed a constant quantity;

$$\therefore\ \frac{1}{a_1}\int_0^{2\pi} P_2 L d\theta=P_2 a_2\times\frac{1}{a_1a_2}\int_0^{2\pi} L d\theta.$$

Now $L=A_1A_2=\{a_1^2+2a_1a_2\cos.\theta+a_2^2\}^{\frac{1}{2}}$;

$$\therefore\ \frac{1}{a_1a_2}\int_0^{2\pi} L d\theta=\frac{1}{a_1a_2}\int_0^{2\pi}(a_1^2+2a_1a_2\cos.\theta+a_2^2)^{\frac{1}{2}}d\theta=$$

$$\frac{(a_1^2+a_2^2)^{\frac{1}{2}}}{a_1a_2}\int_0^{2\pi}\left\{1+\frac{2a_1a_2}{a_1^2+a_2^2}\cos.\theta\right\}^{\frac{1}{2}}d\theta=$$

$$\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right)^{\frac{1}{2}}\int_0^{2\pi}\left\{1+2\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)^{-1}\cos.\theta\right\}^{\frac{1}{2}}d\theta=$$

$$\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right)^{\frac{1}{2}}\int_0^{2\pi}\left\{1+\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)^{-1}\cos.\theta\right\}d\theta \text{ nearly,}$$

neglecting powers of $\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)^{-1}$ above the first, since in all cases its value is less than unity. Integrating this quantity between the limits 0 and 2π the second term disappears, so that

$$\frac{1}{a_1a_2}\int_0^{2\pi}\mathrm{L}d\theta=\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right)^{\frac{1}{2}}2\pi \text{ nearly};$$

$$\therefore \mathrm{P}_2a_2\cdot\frac{1}{a_1a_2}\int_0^{2\pi}\mathrm{L}d\theta=\mathrm{P}_2(2\pi a_2)\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right)^{\frac{1}{2}}=\mathrm{U}_2\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right)^{\frac{1}{2}};$$

since $2\pi a_2$ is the space through which the point of application of the constant pressure P_2 is made to move in each revolution. Therefore by equation (192), in the case in which P_2 is constant,

$$\mathrm{U}_1=\mathrm{U}_2\left\{1+\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right)^{\frac{1}{2}}\rho\sin.\phi\right\}\quad\ldots\ldots(193).$$

181. If the pressure P_2 be supplied by the tension of a rope winding upon a drum whose radius is a_2 (as in the capstan), then is the effect of the rigidity of the rope (Art. 142.) the same as though P_2 were increased by it so as to become

$$\mathrm{P}_2+\frac{\mathrm{D}+\mathrm{EP}_2}{a_2},\text{ or }\left(1+\frac{\mathrm{E}}{a_2}\right)\mathrm{P}_2+\frac{\mathrm{D}}{a_2}.$$

Now, assuming P_2 to be constant, and observing that $\mathrm{U}_2=2\pi\mathrm{P}_2a_2$, we have, by equation (192),

$$\mathrm{U}_1=\mathrm{P}_2a_2\left\{2\pi+\frac{\rho\sin.\phi}{a_1a_2}\int_0^{2\pi}\mathrm{L}d\theta\right\}.$$

Substituting in this equation the *above* value for P_2,

$$\mathrm{U}_1=a_2\left\{\left(1+\frac{\mathrm{E}}{a_2}\right)\mathrm{P}_2+\frac{\mathrm{D}}{a_2}\right\}\left\{2\pi+\frac{\rho\sin.\phi}{a_1a_2}\int_0^{2\pi}\mathrm{L}d\theta\right\}.$$

Performing the actual multiplication of these factors, observing that $\frac{\mathrm{D}}{a_2}$ is exceedingly small, and omitting the term

involving the product of this quantity and $\frac{\rho \sin. \varphi}{a_1}$, we have

$$U_1 = P_2 a_2 \left(1 + \frac{E}{a_2}\right) \left\{ 2\pi + \frac{\rho \sin. \varphi}{a_1 a_2} \int_0^{2\pi} L d\theta \right\} + 2\pi D.$$

Whence performing the integration as before, we obtain

$$U_1 = U_2 \left(1 + \frac{E}{a_2}\right) \left\{ 1 + \left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)^{\frac{1}{2}} \rho \sin. \varphi \right\} + 2\pi D.$$

If this equation be multiplied by n, and if instead of U_1 and U_2 representing the work done during *one* complete revolution, they be taken to represent the work done through n such revolutions, then

$$U_1 = U_2 \left(1 + \frac{E}{a_2}\right) \left\{ 1 + \left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)^{\frac{1}{2}} \rho \sin. \varphi \right\} + 2n\pi D. \ldots (194),$$

which is the MODULUS.

182. If the quantity $\left(\frac{a_1}{a_2} + \frac{a_2}{a_1}\right)^{-1}$ be *not* so small that terms of the binomial expansion involving powers of that quantity above the first may be neglected, the value of the definite integral $\int_0^{2\pi} L d\theta$ may be determined as follows: —

$$\int_0^{2\pi} (a_1^2 + 2a_1 a_2 \cos. \theta + a_2^2)^{\frac{1}{2}} d\theta = \int_0^{2\pi} \{(a_1 + a_2)^2 - 2a_1 a_2 (1 - \cos. \theta)\}^{\frac{1}{2}} d\theta$$

$$= (a_1 + a_2) \int_0^{2\pi} \left\{ 1 - \frac{4a_1 a_2}{(a_1 + a_2)^2} \sin.^2 \frac{\theta}{2} \right\}^{\frac{1}{2}} d\theta. \quad \text{Let } k^2 = \frac{4a_1 a_2}{(a_1 + a_2)^2};$$

$$\therefore \int_0^{2\pi} L d\theta = (a_1 + a_2) \int_0^{2\pi} \left(1 - k^2 \sin.^2 \frac{\theta}{2}\right)^{\frac{1}{2}} d\theta = (a_1 + a_2) \int_0^{\pi} (1 - k^2 \sin.^2 \theta)^{\frac{1}{2}}$$

$$= 2(a_1 + a_2) \int_0^{\frac{\pi}{2}} (1 - k^2 \sin.^2 \theta)^{\frac{1}{2}} d\theta^* = 2(a_1 + a_2) E_1(k), \text{ where } E_1(k)$$

* See *Encyc. Met.* art. DEF. INT. theorem 2.

$$\therefore \int_0^\theta \mathrm{RL}d\theta = \cdot 96 a_1 \left\{ \frac{1}{a_3} \int_0^\theta \mathrm{P}_3 a_3 d\theta + \frac{1}{a_2} \int_0^\theta \mathrm{P}_2 a_2 d\theta \right\} - \int_0^\theta (\mathrm{P}_3 a_3 - \mathrm{P}_2 a_2)(\cdot 96 \cos.\theta - \cdot 4 \sin.\theta)$$

$$\int_0^\theta \mathrm{RL}d\theta = \cdot 96 a_1 \left\{ \frac{\mathrm{U}_3}{a_3} + \frac{\mathrm{U}_2}{a_2} \right\} - \int_0^\theta (\mathrm{P}_3 a_3 - \mathrm{P}_2 a_2)(\cdot 96 \cos.\theta - \cdot 4 \sin.\theta) d\theta. \quad (200).$$

If P_2 and P_3 be *constant*, the integral in the second member of this equation becomes $(\mathrm{P}_3 a_3 - \mathrm{P}_2 a_2)(\cdot 96 \sin.\theta + \cdot 4 \cos.\theta)$; whence observing that $\mathrm{P}_3 a_3 - \mathrm{P}_2 a_2 = \frac{\mathrm{P}_3 a_3 \theta - \mathrm{P}_2 a_2 \theta}{\theta} = \frac{\mathrm{U}_3 - \mathrm{U}_2}{\theta}$; also, that $\mathrm{U}_r = \theta \mathrm{R} r = \theta \mathrm{P}_3 a_3 - \theta \mathrm{P}_2 a_2 = \mathrm{U}_3 - \mathrm{U}_2$, and substituting in equation (198), we have

$$\mathrm{U}_1 = \mathrm{U}_3 - \mathrm{U}_2 + \rho \sin.\phi \left\{ \cdot 96 \left(\frac{\mathrm{U}_3}{a_3} + \frac{\mathrm{U}_2}{a_2} \right) - \left(\frac{\mathrm{U}_3 - \mathrm{U}_2}{a_1 \theta} \right) (\cdot 96 \sin.\theta + \cdot 4 \cos.\theta \right\} .. (201);$$

for a complete revolution making $\theta = 2\pi$, we have

$$\mathrm{U}_1 = \mathrm{U}_3 - \mathrm{U}_2 + \rho \sin.\phi \left\{ \cdot 96 \left(\frac{\mathrm{U}_3}{a_3} + \frac{\mathrm{U}_2}{a_2} \right) - \cdot 4 \left(\frac{\mathrm{U}_3 - \mathrm{U}_2}{2\pi a_1} \right) \right\};$$

reducing,

$$\mathrm{U}_1 = \left\{ 1 + \frac{\rho \sin.\phi}{5} \left(\frac{4 \cdot 8}{a_3} - \frac{1}{a_1 \pi} \right) \right\} \mathrm{U}_3 - \left\{ 1 - \frac{\rho \sin.\phi}{5} \left(\frac{4 \cdot 8}{a_2} + \frac{1}{a_1 \pi} \right) \right\} \mathrm{U}_2 . \; .. (202),$$

which is the modulus of the system.

185. If the pressure P_3 be supplied by the tension of a cord which winds upon a cylinder or drum at the point A_3, then allowance must be made for the rigidity of the cord, and a correction introduced into the preceding equation for that purpose. To make this correction let it be observed (Art. 142.) that the effect of the rigidity of the cord at A_3 is the same as though it increased the tension there from

$$\mathrm{P}_3 \text{ to } \mathrm{P}_3 \left(1 + \frac{\mathrm{E}}{a_3} \right) + \frac{\mathrm{D}}{a_3};$$

or (multiplying both sides of this inequality by a_3, and integrating in respect to θ,) as though it increased

$$\int_0^{2\pi} \mathrm{P}_3 a_3 d\theta \text{ to } \left(1 + \frac{\mathrm{E}}{a_3} \right) \int_0^{2\pi} \mathrm{P}_3 a_3 d\theta + \int_0^{2\pi} \mathrm{D} d\theta;$$

[redacted]

$$\therefore a_1\int_0^{2n\pi}(p_1+p\ )d\theta = a_1\{W+P_2+2w\}2n\pi = \left(\frac{a_1}{a_2}\right)\{(2n\pi a_2)P_2+(2n\pi a_2)(W+2w)\}$$

$$= \left(\frac{a_1}{a_2}\right)\{S_2P_2+S_2(W+2w)\} = \left(\frac{a_1}{a_2}\right)\{U_2+S_2(W+2w)\};$$

representing by S_2 the space described by the load, and by U_2 the useful work done upon it, during n revolutions of the capstan.

Similarly,

$$a_2\int_0^{2n\pi}(p_3-p_2)(\cdot 96\cos.\theta-\cdot 4\sin.\theta)d\theta = a_2\int_0^{2n\pi}\{W+P_2-2\mu a_2\theta\}(\cdot 96\cos.\theta-\cdot 4\sin.\theta)d\theta$$

$$= a_2(W+P_2)\int_0^{2n\pi}(\cdot 96\cos.\theta-\cdot 4\sin.\theta)d\theta-2\mu a_2^2\int(\cdot 96\cos.\theta-\cdot 4\sin.\theta)\theta d\theta.$$

Now $\int_0^{2n\pi}(\cdot 96\cos.\theta-\cdot 4\sin.\theta)d\theta=\cdot 4$, and $\int_0^{2n\pi}(\cdot 96\cos.\theta-\cdot 4\sin.\theta)\theta d\theta=\cdot 8n\pi$ *;

$$\therefore a_2\int_0^{2n\pi}(p_3-p_2)(\cdot 96\cos.\theta-\cdot 4\sin.\theta)d\theta=\cdot 4a_2(W+P_2)-\cdot 8\mu a_2(2n\pi a_2)$$

$$=\cdot 4a_2\frac{U_2}{S_2}+\cdot 4a_2(W-2\mu S_2);\text{ observing that } P_2=\frac{U_2}{S_2},$$

$$\therefore\int_0^{2n\pi}RLd\theta=\cdot 96\left(\frac{a_1}{a_2}\right)\{U_2+S_2(W+2w)\}-\cdot 4a_2\frac{U_2}{S_2}-\cdot 4a_2(W-2\mu S_2);$$

$$\frac{\rho\sin.\phi}{a_1}\int_0^{2n\pi}RLd\theta=\frac{\cdot 4\rho\sin.\phi}{a_1}\left\{\left(2\cdot 4\frac{a_1}{a_2}-\frac{a_2}{S_2}\right)U_2+2\left\{1\cdot 2(W+2w)\left(\frac{a_1}{a_2}\right)+\mu a_2\right\}S-Wa_2\right\}.$$

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* For $\int_0^\theta\theta\cos.\theta d\theta=\theta\sin.\theta-\int_0^\theta\sin.\theta d\theta=\theta\sin.\theta-\text{vers.}\,\theta$; also $\int_0^\theta\theta\sin.\theta d\theta$ $=-\theta\cos.\theta+\int_0^\theta\cos.\theta d\theta=-\theta\cos.\theta+\sin.\theta$. Now, substituting $2n\pi$ for θ, these integrals become respectively 0 and $-2n\pi$.

exceedingly numerous. By this supposition we shall manifestly approach exceedingly near to the actual case of an *infinite* number of such points and a *continuous* surface; and if we suppose $\Delta\theta$ infinitely small, our supposition will *coincide* with that case. Now, on the supposition that $\Delta\theta$ is exceedingly small, $\tan.\frac{\Delta\theta}{2}.\tan.\varphi$ is exceedingly small, and may be neglected as compared with unity; it may therefore be neglected in the denominator of the above fraction. Moreover, $\Delta\theta$ being exceedingly small, $\tan.\frac{\Delta\theta}{2}=\frac{\Delta\theta}{2}$

$$\therefore \frac{T_1-T}{T_2}=\tan.\varphi.\Delta\theta^*; \therefore T_1=T_2(1+\tan.\varphi.\Delta\theta).$$

Now the number of the points A, B, C, &c. being represented by n, and the whole angle AdZ between the *extreme* normals at A and Z by θ, it follows (Euclid, i. 32.) that $\theta=n.\Delta\theta$; therefore $\Delta\theta=\frac{\theta}{n}$;

$$\therefore T_1=T_2\left(1+\frac{\theta}{n}\tan.\varphi\right).$$

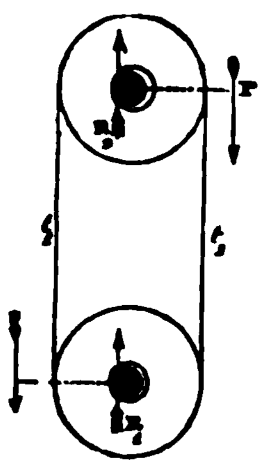
Similarly,
$$P_1=T_1\left(1+\frac{\theta}{n}\tan.\varphi\right),$$

$$T_2=T_3\left(1+\frac{\theta}{n}\tan.\varphi\right),$$

$$\&c.=\&c.=\&c.$$

$$T_{n-1}=P_2\left(1+\frac{\theta}{n}\tan.\varphi\right).$$

* If we consider the tension T as a function of θ, of which any consecutive values are represented by T_1 and T_2, and their difference or the increment of T by ΔT, then $\frac{-\Delta T}{T}=\tan.\phi.\Delta\theta$; therefore $\frac{1}{T}\cdot\frac{\Delta T}{\Delta\theta}=-\tan.\phi$; therefore, passing to the *limit* $\frac{1}{T}\frac{dT}{d\theta}=-\tan.\phi$, and *integrating* between the limits 0 and θ, observing that at the latter limit $T=P_2$, and that at the former it equals P_1, we have $\log._\epsilon\left(\frac{P_2}{P_1}\right)=-\theta\tan.\phi$; therefore $P_1=P_2\epsilon^{\theta\tan.\phi}$.



action of the machine, be opposed to the motion of the drum to be represented by a pressure P applied at a given distance a from its centre C_2. Suppose, moreover, that the band has received such an initial tension T as shall just cause it to be on the point of slipping when the motion of the drum is subjected to this maximum resistance; and let t_1 and t_2 be the tensions upon the two parts of the band when it is thus just in the act of slipping and of overcoming the resistance P. Now, the two parts of the band being parallel, it embraces one half of the circumference of each drum; the relation between t_1 and t_2 is therefore expressed (equation 210) by the equation $t_1=t_2\varepsilon^{\pi \tan. \phi}$, whence we obtain $\frac{t_1-t_2}{t_1+t_2}=\frac{\varepsilon^{\pi \tan. \phi}-1}{\varepsilon^{\pi \tan. \phi}+1}$. But $t_1+t_2=$ 2T (equation 214),

$$\therefore\ t_1-t_2=2\mathrm{T}\left(\frac{\varepsilon^{\pi \tan. \phi}-1}{\varepsilon^{\pi \tan. \phi}+1}\right).$$

Also, the relation between the resistance P, opposed to the motion of the upper drum, and the tensions t_1 and t_2 upon the two parts of the band, when this resistance is on the the point of being overcome, is expressed (equation 217) by the equation

$$\mathrm{P}a+t_2r+\mathrm{R}_2\rho \sin. \phi=t_1r;$$

or substituting the value of R_2 (equation 216), and transposing,

$$\mathrm{P}a+(2\mathrm{T}\mp\mathrm{P}+\mathrm{W})\rho \sin. \phi=(t_1-t_2)r;$$

whence, substituting the value of t_1-t_2, determined above, and transposing, we have

$$\mathrm{P}(a\mp\rho \sin. \phi)+\mathrm{W}\rho \sin. \phi=2\mathrm{T}\left\{\left(\frac{\varepsilon^{\pi \tan. \phi}-1}{\varepsilon^{\pi \tan. \phi}+1}\right)r-\rho \sin. \phi\right\};$$

The number of units of work transmitted by any machine per minute is usually represented in horses' power, one horse's power being estimated at 33,000 units, so that the number of horses' power transmitted by the machine means the number of times 33,000 units of work are transmitted by it in each minute, or the number of times 33,000 must be taken to equal the number of units of work transmitted by it each minute. If therefore H represent the number of horses' power transmitted by the wheel, then $U=33,000H$. Substituting this value in the preceding equation, and representing the constant $33,000c^2$ by C^3, we have

$$T=C\sqrt[3]{\frac{H}{mn}} \quad \ldots\ldots \quad (239).$$

The values of the constant C for teeth of different materials are given in the Appendix.

209. *To determine the radius of the pitch circle of a wheel which shall contain n teeth of a given pitch.*

Let AB represent the pitch T of a tooth, and let it be supposed to coincide with its chord AMB. Let R represent the radius AC of the pitch circle, and n the number of teeth to be cut upon the wheel.

Now there are as many pitches in the circumference as teeth, therefore the angle ACB subtended by each pitch is represented by $\frac{2\pi}{n}$.

And $T=2AM=2AC \sin. \frac{1}{2}ACB=2R \sin. \frac{\pi}{n}$;

$$\therefore\ R=\tfrac{1}{2}T \operatorname{cosec.}\frac{\pi}{n} \quad \ldots\ldots \quad (240).$$

Substituting Nr_1r_2 for the factor, which it represents in equation (250), we have

$$P_1a_1\{r_2\sin.(\theta+\phi)-\lambda\sin.\phi-\frac{L_2\rho_2}{a_2}\sin.\phi_2\}-P_2a_2\{r_1\sin.(\theta+\phi)+\lambda\sin.\phi+\frac{L_1\rho_1}{a_1}\sin.\phi_1\}=Nr_1r_2\sin.(\theta+\varphi)\ \ldots\ldots\ (253).$$

Solving this equation in respect to P_1,

$$P_1=\frac{a_2r_1}{a_1r_2}\left\{\frac{1+\dfrac{\lambda\sin.\varphi+\dfrac{L_1\rho_1}{a_1}\sin.\varphi_1}{r_1\sin.(\theta+\varphi)}}{1-\dfrac{\lambda\sin.\varphi+\dfrac{L_2\rho_2}{a_2}\sin.\varphi_2}{r_2\sin.(\theta+\varphi)}}\right\}P_2+\frac{\dfrac{Nr_1}{a_1}}{1-\dfrac{\lambda\sin.\varphi+\dfrac{L_2\rho_2}{a_2}\sin.\varphi_2}{r_2\sin.(\theta+\varphi)}}.$$

Whence, performing actual division by the denominators of the fractions in the second member of the equation, and omitting terms of two dimensions in $\sin.\varphi_1$, $\sin.\varphi_2$, $\sin.\varphi$ (observing that N is already of *one* dimension in those variables), we have

$$P_1=\frac{a_2r_1}{a_1r_2}\left\{1+\{\lambda\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\sin.\phi+\frac{L_1\rho_1}{a_1r_1}\sin.\phi_1+\frac{L_2\rho_2}{a_2r_2}\sin.\phi_2\}\operatorname{cosec.}(\theta+\phi)\right\}P_2+\frac{Nr_1}{a_1}\ \ldots\ (2$$

In this expression it is assumed that the contact of the teeth is behind the line of centres.

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this case, equal to r_1, and the point M is supposed to coincide with A, L_1 becomes a chord of the pitch circle, and is therefore represented by $2r_1\sin.\frac{1}{2}$DBA, or $2r_1\sin.\frac{1}{2}(\alpha_1+\beta)$; so that $\dfrac{\sin.\alpha_1+\dfrac{a_1}{r_1}\sin.\beta}{L_1}=$

$$\frac{\sin.\alpha_1+\sin\beta}{2r_1\sin.\frac{1}{2}(\alpha_1+\beta)}=\frac{2\sin.\frac{1}{2}(\alpha_1+\beta)\cos.\frac{1}{2}(\alpha_1+\beta)}{2r_1\sin.\frac{1}{2}(\alpha_1+\beta)}=\frac{1}{r_1}\cos.\tfrac{1}{2}(\alpha_1+\beta).$$

If, therefore, we take the angle $\alpha_1=-\beta$, so as to give to P_1 the direction of a tangent at A, this expression will assume the value, $\frac{1}{r_1}\cos.0$, or $\frac{1}{r_1}$; so that in this case

$$N=\frac{W_1\rho_1}{r_1}\sin.\phi_1-\frac{W_2\rho_2}{L_2}(\sin.\alpha_2+\frac{a_2}{r_2}\sin.\beta)\sin.\phi_2.$$

221. The modulus of a system of two toothed wheels.

Let n_1 and n_2 represent the numbers of teeth in the driving and driven wheels respectively, and let it be observed that these numbers are one to another as the radii of the pitch circles of the wheels; then, multiplying both sides of equation (254) by $a_1\frac{r_2}{r_1}$, we shall obtain

$$P_1a_1\frac{r_2}{r_1}=P_2a_2\left\{1+\{\lambda\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\sin.\phi+\frac{L_1\rho_1}{a_1r_1}\sin.\phi_1+\frac{L_2\rho_2}{a_2r_2}\sin.\phi_2\}\operatorname{cosec.}(\theta+\phi)\right\}+Nr_2.$$

Now let $\Delta\psi$ represent an exceedingly small increment of the angle ψ, through which the driven wheel is supposed to have revolved, after the point of contact P has passed the line of centres; and let it be observed that the first member of the above equation is equal to $P_1a_1\frac{r_2}{r_1}\frac{\Delta\psi}{\Delta\psi}$, and that $\frac{r_2}{r_1}\Delta\psi$ represents the angle described by the driving wheel (Art. 206), whilst the driven wheel describes the angle $\Delta\psi$; whence it follows (Art. 50.) that $P_1a_1\left(\frac{r_2}{r_1}\Delta\psi\right)$ represents the *work* ΔU_1 done by the driving pressure P_1, whilst this angle $\Delta\psi$ is described by the driven wheel,

$$\frac{\Delta U}{\Delta\psi}=P_2a_2\left\{1+\{\lambda\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\sin.\phi+\frac{L_1\rho_1}{a_1r_1}\sin.\phi_1+\frac{L_2\rho_2}{a_2r_2}\sin.\phi_2\}\operatorname{cosec.}(\theta+\phi)\right\}+Nr_2.$$

Let now $\Delta\psi$ be conceived infinitely small, so that the first member of the above equation may become the differential co-efficient of U_1, in respect to ψ. Let the equation, then, be integrated between the limits 0 and ψ; P_2, L_1, and L_2, and therefore N (equation 252) being conceived to remain constant, whilst the angle ψ is described; we shall then obtain the equation

$$U_1=P_2a_2\int_0^{\psi}\left\{1+\{\lambda\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\sin.\phi+\frac{L_1\rho_1}{a_1r_1}\sin.\phi_1+\frac{L_2\rho_2}{a_2r_2}\sin.\phi_2\}\operatorname{cosec.}(\theta+\phi)\right\}d\psi+N\cdot S\ldots(255),$$

remain constant, whilst any two given teeth are in action, $P_2 a_2 \psi$ represents the work U_2 yielded by that pressure whilst those teeth are in contact: also $r_2 \psi$ represents the space S, described by the circumference of the pitch circle of either wheel whilst this angle is described. Now let ψ be conceived to represent the angle subtended by the pitch of one of the teeth of the driven wheel, these teeth being supposed to act only *behind* the line of centres, then $\psi = \frac{2\pi}{n_2}$, n_2 representing the number of teeth on the driven wheel,

$$\text{and } \tfrac{1}{2}\psi\left(1+\frac{n_2}{n_2}\right)=\frac{\pi}{n_2}\left(1+\frac{n_2}{n_1}\right)=\pi\left(\frac{1}{n_1}+\frac{1}{n_2}\right);$$

$$\therefore U_1=\left\{1+\pi\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\sin.\phi+\frac{L_1\rho_1}{a_1 r_1}\sin.\phi_1+\frac{L_2\rho_2}{a_2 r_2}\sin.\phi_2\right\}U_2+N.S \quad . \; . \; . \quad (257),$$

which relation between the work done at the moving and working points, whilst any two given teeth are in contact, is evidently also the relation between the work similarly done, whilst *any given number* of teeth are in contact. It is therefore the MODULUS of any system of two toothed wheels, the numbers of whose teeth are considerable.

223. THE MODULUS OF A SYSTEM OF TWO WHEELS WITH INVOLUTE TEETH OF ANY NUMBERS AND DIMENSIONS.

The locus of the points of contact of the teeth has been shown (Art. 203.) to be in this case a straight line DE, which passes through the point of contact A of the pitch circles, and touches the circles (EF and DG) from which the involutes are struck. Let P represent any position of this point of contact, then is AP measured along the given line DE the distance represented by λ in Art. 218., and the angle CAD, which is in this case constant, is that represented

the angle ACD, or its equal the angle DAI, which the tangent DE to the circles from which the involutes are struck makes with a perpendicular AI to the line of centres. Moreover, that the co-efficient N not involving this factor η (equation 252), the variation of the value of U_1, so far as this angle is concerned, is wholly involved in the corresponding variation of the co-efficient of U_2 and becomes a minimum with it; so that the value of η which gives to the function of η represented by this co-efficient, its minimum value, is the value of it which satisfies the condition of the *greatest economy of power*, and determines that inclination DAI of the tangent DE to the perpendicular to the line of centres, and those values, therefore, of the radii CD and BE of the circles whence the involutes are struck, which correspond to the tooth of least resistance.

To determine the value of η which corresponds to a minimum value of this co-efficient, let the latter be represented by u; then, for the required value of η,

$$\frac{du}{d\eta}=0, \text{ and } \frac{d^2u}{d\eta^2}>0.$$

$$\text{Let } \pi\left(\frac{1}{n_1}+\frac{1}{n_2}\right)=\text{A}, \quad \frac{\text{L}_1\rho_1}{a_1r_1}\sin.\varphi_1+\frac{\text{L}_2\rho_2}{a_2r_2}\sin.\varphi_2=\text{B};$$

$$\therefore\ u=1+(\text{A}\cos.\eta\sin.\varphi+\text{B})\sec.(\eta-\phi);$$

$$\therefore\ u=1+\text{B}\sec.(\eta-\varphi)+\text{A}\sin.\varphi\cos.\eta\sec.(\eta-\varphi);$$

$$\frac{du}{d\eta}=\text{B}\sec.(\eta-\phi)\tan.(\eta-\phi)-\text{A}\sin.\phi\{\sin.\eta\sec.(\eta-\phi)-\cos.\eta\tan.(\eta-\phi)\sec.(\eta-\phi)\}$$

$$=\text{B}\sec.^2(\eta-\phi)\sin.(\eta-\phi)-\text{A}\sin.\phi\sec.^2(\eta-\phi)\{\sin.\eta\cos.(\eta-\phi)-\cos.\eta\sin.(\eta-\phi)\}$$

$$\therefore\ \frac{du}{d\eta}=\sec.^2(\eta-\varphi)\{\text{B}\sin.(\eta-\varphi)-\text{A}\sin.^2\varphi\}\ \ldots\ldots\ (260).$$

In order, therefore, that $\frac{du}{d\eta}$ may vanish for any value of η, one of the factors which compose the second member of the above equation must vanish for that value of η; but this can never be the case in respect to the first factor, for the least value of the square of the secant of an arc is the square of

of the pitch circle of the driven wheel to the radius of the *generating* circle. Now the chord $AP=2\overline{AD}$ sin. $\frac{1}{2}$ ADP; therefore $\lambda=2r$ sin. $e\psi=\frac{r_2}{e}$ sin. $e\psi$. Substituting this value of λ in equation (255); observing, moreover, that the angle PAD represented by θ in that equation is equal to $\frac{\pi}{2}-\frac{1}{2}$ ADP, or to $\frac{\pi}{2}-e\psi$, and that the *whole* angle ψ through which the driven wheel is made to revolve by the contact of each of its teeth is represented by $\frac{2\pi}{n_2}$, we have

$$U_1=Pa_1\int_0^{\frac{2\pi}{n_2}}\left\{1+\left\{\frac{r_2}{e}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\sin.\phi\,\sin.e\psi+\frac{L_1\rho_1}{a_1r_1}\sin.\phi_1+\frac{L_2\rho_2}{a_2r_2}\sin.\phi_2\right\}\sec.(e\psi-\phi)\right\}d\psi+NS;$$

or, assuming L_1 and L_2 to remain constant during the contact of any two teeth representing the constant $1+\frac{L_1\rho_1}{a_1r_1}\sin.\phi_1+\frac{L_2\rho_2}{a_2r_2}\sin.\phi_2$ by A, and observing that $\frac{r_2}{r_1}=\frac{n_2}{n_1}$,

$$U_1=Pa_1\left\{A\int_0^{\frac{2\pi}{n_2}}\sec.(e\psi-\phi)d\psi+\frac{1}{e}\left(1+\frac{n_2}{n_1}\right)\sin.\phi\int_0^{\frac{2\pi}{n_2}}\sin.e\psi\,\sec.(e\psi-\phi)d\psi\right\}+NS.$$

Now the *general* integral, $\int\sec.(e\psi-\phi)d\psi$, or $\frac{1}{e}\int\sec.(e\psi-\phi)\,d(e\psi-\phi)$ being represented* by the function $\frac{1}{e}\log._\epsilon\tan.\left\{\frac{\pi}{4}+\frac{1}{2}(e\psi-\phi)\right\}$, its *definite* integral between the limits 0 and $\frac{2\pi}{n_2}$ has for its expression,

$$\frac{1}{e}\log._\epsilon\frac{\tan.\left\{\frac{\pi}{4}+\frac{1}{2}\left(e\frac{2\pi}{n_2}-\phi\right)\right\}}{\tan.\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}.$$

* Hymer's Integ. Cal. Art. 52.

[redacted]

In like manner it may be shown that $\sin.\gamma_2=\cos.(\theta+\varphi)\sin.\iota_2$, being taken to represent the inclination KAL of the nes AE and AK, which angle is also equal to the angle CG.

From the above equations it follows that

$$\left.\begin{array}{l} =R\cos.\gamma_1=R\sqrt{1-\cos.^2(\theta+\varphi)\sin.^2\iota_1} \\ =R\cos.\gamma_2=R\sqrt{1-\cos.^2(\theta+\varphi)\sin.^2\iota_2} \end{array}\right\} \quad \ldots\ldots \quad (273).$$

From the centre b of the circle AD draw bm per-

ndicular to RA, then is BM (the perpendicular let fall m the centre of the circle AH upon the direction of R_1) e *projection* of bm upon the plane of the circle AH. To termine the inclination of bm to the plane AH, draw An rallel to bm; the sine of the inclination of An to the ne AH is then determined to be $\cos.DAn.\sin.\iota_1$, precely as the sine of the inclination of Am to the same ne was before determined to be $\cos.DAm.\sin.\iota_1$.

Now $DAn=Abm=\frac{\pi}{2}-DAR=\frac{\pi}{2}-(\theta+\varphi)$; therefore the e of the inclination of An, and therefore of bm, to the plane H is represented by the formula $\sin.(\theta+\varphi)\sin.\iota_1$, and the ine of its inclination by $\sqrt{1-\sin.^2(\theta+\varphi)\sin.^2\iota_1}$;

$$\therefore\ m_1=BM=\overline{bm}\sqrt{1-\sin.^2(\theta+\varphi)\sin.^2\iota_1}.$$

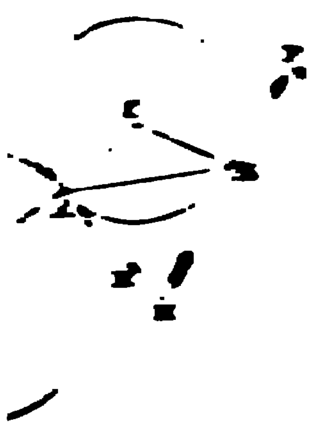
$$\ldots - \frac{\cos.\,\theta_2}{a_2}\Big) \cos.\,\phi_2 \sin.\,\phi + \frac{\rho_1 \sin.\,\phi_1}{ba_1}\int_0^b \frac{L}{r}dx + \frac{\rho_2 \sin.\,\phi_2}{ba_2}\int_0^b \frac{L'}{r'}dx\Big\}. \quad . \text{(277).}$$

$a-x$ being taken to represent the distance of the contact of any two such elements from C, and a to the distance CF, the radii r and r' of these elere evidently (by similar triangles) represented by

$$\frac{a+x}{a}r_1 \text{ or } \left(1+\frac{x}{a}\right)r_1, \text{ and } \frac{a+x}{a}r_2 \text{ or } \left(1+\frac{x}{a}\right)r_2,$$

r_1 and r_2 reg the radii of the extreme elements NF and OF, or ich circles of the lesser extremities of the wheels.

ssuming, as we have done, the pressures R_1 and R_2 rpendicular to the lines BA, GA joining the centre



of each element with their point of contact A, so that the points M and N (see *fig.* p. 315.) coincide with the point A (see accompanying figure)*; and representing the angles ABD and ACE made by the perpendiculars DB and CE with the line of centres by θ_1 spectively; observing also that $\overline{AD}^2 = \overline{BA}^2 - 2\overline{BA}.$

$\overline{ABD} + \overline{BD}^2$, so that $\left(\frac{AD}{BA}\right)^2 = 1 - 2\left(\frac{BD}{BA}\right)\cos.\,ABD$

, we have, substituting, in the second member of tion, for BA or r its value $r_1\left(1+\frac{x}{a}\right)$

$$=1-2\left(\frac{a_1}{r_1}\right)\left(1+\frac{x}{a}\right)^{-1}\cos.\,\theta_1 + \left(\frac{a_1}{r_1}\right)^2\left(1+\frac{x}{a}\right)^{-2};$$

nding the binomials in this expression, observing an exceedingly small quantity, neglecting terms inowers of that quantity above the first, and reducing,

* ircles in this figure represent two of the corresponding laminæ wheels have been imagined to be divided; they are not, there- same plane. Their planes intersect in AH.

$$\left(\frac{\mathrm{L}}{r}\right)^2=1-2\left(\frac{a_1}{r_1}\right)\cos.\theta_1+\left(\frac{a_1}{r_1}\right)^2+2\left(\frac{a_1}{r_1}\right)\left(\cos.\theta_1-\frac{a_1}{r_1}\right)\frac{x}{a}\ .\ (278).$$

Now L_1 representing the value of L when $x=0$, and θ remaining constant,

$$\left(\frac{\mathrm{L}_1}{r_1}\right)^2=1-2\left(\frac{a_1}{r_1}\right)\cos.\theta_1+\left(\frac{a_1}{r_1}\right)^2;$$

$$\therefore\ 2\left(\frac{a_1}{r_1}\right)\left(\cos.\theta_1-\frac{a_1}{r_1}\right)=1-\left(\frac{a_1}{r_1}\right)^2-\left(\frac{\mathrm{L}_1}{r_1}\right)^2.$$

Let now the angle ADB, made in respect to the first element of the driving wheel between the perpendicular BD or a_1 and the chord AD or L, be represented by η_1, and let η_2 represent the corresponding angle in the driven wheel, then

$$\mathrm{L}_1^2-2\mathrm{L}_1a_1\cos.\eta_1+a_1^2=r_1^2,\ \therefore\ \left(\frac{\mathrm{L}_1}{r_1}\right)^2-2\frac{\mathrm{L}_1a_1}{r_1^2}\cos.\eta_1+\left(\frac{a_1}{r_1}\right)^2=1;$$

$$\therefore\ -2\frac{\mathrm{L}_1a_1}{r_1^2}\cos.\eta_1=1-\left(\frac{a_1}{r_1}\right)^2-\left(\frac{\mathrm{L}_1}{r_1}\right)^2=2\left(\frac{a_1}{r_1}\right)\left(\cos.\theta_1-\frac{a_1}{r_1}\right).$$

Substituting these values of $\left(\frac{\mathrm{L}_1}{r_1}\right)^2$ and $2\left(\frac{a_1}{r_1}\right)\left(\cos.\theta-\frac{a_1}{r_1}\right)$ in equation (278);

$$\left(\frac{\mathrm{L}}{r}\right)^2=\left(\frac{\mathrm{L}_1}{r_1}\right)^2-2\left(\frac{\mathrm{L}_1a_1}{r_1^2}\right)\left(\frac{x}{a}\right)\cos.\eta_1=\left(\frac{\mathrm{L}_1}{r_1}\right)^2\left\{1-2\left(\frac{a_1}{\mathrm{L}_1}\right)\left(\frac{x}{a}\right)\cos.\eta_1\right\};$$

Extracting the square root of the binomial, and neglecting terms involving powers of $\frac{x}{a}$ above the first,

$$\frac{\mathrm{L}}{r}=\frac{\mathrm{L}_1}{r_1}-\left(\frac{a_1}{r_1}\right)\left(\frac{x}{a}\right)\cos.\eta_1=\frac{a_1}{r_1}\left\{\frac{\mathrm{L}_1}{a_1}-\frac{x}{a}\cos.\eta_1\right\};$$

$$\because\ \text{(Equation 277)}\ \frac{\rho_1\sin.\phi_1}{ba_1}\int_0^b\frac{\mathrm{L}}{r}dx=\frac{\rho_1\sin.\phi_1}{r_1}\left\{\frac{\mathrm{L}_1}{a_1}-\tfrac{1}{2}\frac{b}{a}\cos.\eta_1\right\}.$$

$$\text{Similarly}\ \frac{\rho_2\sin.\phi_2}{ba_2}\int_0^b\frac{\mathrm{L}'}{r'}dx=\frac{\rho_2\sin.\phi_2}{r_2}\left\{\frac{\mathrm{L}_2}{a_2}-\tfrac{1}{2}\frac{b}{a}\cos.\eta_2\right\}.$$

Substituting these values in the modulus (equation 277),

$$U_1 = U_2\left\{1+\pi\left(\frac{\cos.\iota_1}{n_1}+\frac{\cos.\iota_2}{n_2}\right)\cos.\iota_2\sin.\phi+\frac{\rho_1\sin.\phi_1}{r_1}\left(\frac{L_1}{a_1}-\frac{1}{2}\frac{b}{a}\cos.\eta_1\right)+\right.$$

$$\left.\frac{\rho_2\sin.\phi_2}{r_2}\left(\frac{L_2}{a_2}-\frac{1}{2}\frac{b}{a}\cos.\eta_2\right).\right\}$$

Now let the angle BCG, or the inclination of the axes, from one to the other of which motion is transferred by the wheels, be represented by 2ι; therefore $\iota_1+\iota_2=2\iota$. Also $a\sin.\iota_1=r_1$ and $a\sin.\iota_2=r_2$,

$$\therefore\ \frac{\sin.\iota_1}{\sin.\iota_2}=\frac{r_1}{r_2}=\frac{n_1}{n_2};$$

$$\therefore\ \frac{\sin.^2\iota_1}{n_1^2}=\frac{\sin.^2\iota_2}{n_2^2};\ \therefore\ \frac{1}{n_1^2}-\frac{\cos.^2\iota_1}{n_1^2}=\frac{1}{n_2^2}-\frac{\cos.^2\iota_2}{n_2^2};$$

$$\therefore\ \frac{1}{n_1^2}-\frac{1}{n_2^2}=\frac{\cos.^2\iota_1}{n_1^2}-\frac{\cos.^2\iota_2}{n_2^2}=\left(\frac{\cos.\iota_1}{n_1}+\frac{\cos.\iota_2}{n_2}\right)\left(\frac{\cos.\iota_1}{n_1}-\frac{\cos.\iota_2}{n_2}\right)=$$

$$\left(\frac{\cos.\iota_1}{n_1}+\frac{\cos.\iota_2}{n_2}\right)\left(\frac{1}{n_1}\frac{\cos.\iota_1}{\cos.\iota_2}-\frac{1}{n_2}\right)\cos.\iota_2;$$

$$\therefore\ \left(\frac{\cos.\iota_1}{n_1}+\frac{\cos.\iota_2}{n_2}\right)\cos.\iota_2=\frac{\dfrac{1}{n_1^2}-\dfrac{1}{n_2^2}}{\dfrac{1}{n_1}\dfrac{\cos.\iota_1}{\cos.\iota_2}-\dfrac{1}{n_2}}.$$

Now $\dfrac{\cos.\iota_1}{\cos.\iota_2}=\dfrac{\cos.\{\iota+\frac{1}{2}(\iota_1-\iota_2)\}}{\cos.\{\iota-\frac{1}{2}(\iota_1-\iota_2)\}}=\dfrac{1-\tan.\frac{1}{2}(\iota_1-\iota_2)\tan.\iota}{1+\tan.\frac{1}{2}(\iota_1-\iota_2)\tan.\iota}$;

also $\dfrac{n_1}{n_2}=\dfrac{\sin.\iota_1}{\sin.\iota_2}=\dfrac{\sin.\{\iota+\frac{1}{2}(\iota_1-\iota_2)\}}{\sin.\{\iota-\frac{1}{2}(\iota_1-\iota_2)\}}=\dfrac{\tan.\iota+\tan.\frac{1}{2}(\iota_1-\iota_2)}{\tan.\iota-\tan.\frac{1}{2}(\iota_1-\iota_2)}$

$$\therefore\ \tan.\tfrac{1}{2}(\iota_1-\iota_2)=\frac{n_1-n_2}{n_1+n_2}\tan.\iota;$$

$$\therefore\ \frac{\cos.\iota_1}{\cos.\iota_2}=\frac{1-\dfrac{n_1-n_2}{n_1+n_2}\tan.^2\iota}{1+\dfrac{n_1-n_2}{n_1+n_2}\tan.^2\iota}=\frac{(n_1+n_2)-(n_1-n_2)\tan.^2\iota}{(n_1+n_2)+(n_1-n_2)\tan.^2\iota};$$

$$\frac{1}{n_1}\frac{\cos.\iota_1}{\cos.\iota_2}-\frac{1}{n_2}=\frac{1}{n_1n_2}\left(n_2\frac{\cos.\iota_1}{\cos.\iota_2}-n_1\right)=\frac{-1}{n_1n_2}\frac{(n_1^2-n_2^2)+(n_1^2-n_2^2)\tan.^2\iota}{(n_1+n_2)+(n_1-n_2)\tan.^2\iota}$$

is the same with that described at the same distance from its centre by the second wheel, so that $\frac{S_2}{r_3}=\frac{S_1}{r_2}$; in like manner that the spaces described at distances unity from their centres by the fourth and fifth wheels are the same, so that $\frac{S_3}{r_5}=\frac{S_2}{r_4}$; and similarly, that $\frac{S_4}{r_7}=\frac{S_3}{r_6}$, &c.=&c.; and finally, $\frac{S_p}{r_{2p-1}}=\frac{S_{p-1}}{r_{2p-2}}$.

Multiplying the *two* first of these equations together, then the *three* first, the *four* first, &c., and transposing, we have

$$S_1=\frac{r_1}{a_1}S, \quad S_2=\frac{r_1 \cdot r_3}{a_1 \cdot r_2}S=\left(\frac{r_1}{a_1}\right)\left(\frac{n_3}{n_2}\right)S,$$

$$S_3=\frac{r_1 \cdot r_3 \cdot r_5}{a_1 \cdot r_2 \cdot r_4}S=\left(\frac{r_1}{a_1}\right)\left(\frac{n_3 \cdot n_5}{n_2 \cdot n_4}\right)S,$$

$$S_4=\frac{r_1 \cdot r_3 \cdot r_5 \cdot r_7}{a_1 \cdot r_2 \cdot r_4 \cdot r_6}S=\left(\frac{r_1}{a_1}\right)\left(\frac{n_3 \cdot n_5 \cdot n_7}{n_2 \cdot n_4 \cdot n_6}\right)S,$$

&c.=&c.

$$S_p=\frac{r_1 \cdot r_3 \cdot r_5 \cdots r_{2p-1}}{a_1 \cdot r_2 \cdot r_4 \cdots r_{2p-2}}S=\left(\frac{r_1}{a_1}\right)\left(\frac{n_3 \cdot n_5 \cdots n_{2p-1}}{n_2 \cdot n_4 \cdots n_{2p-2}}\right)S.$$

Substituting these values of S_1, S_2, &c. in equation (283), and dividing by S, we have

$$N=\left(\frac{r_1}{a_1}\right)\left\{N_1+(1+\mu_1)\left(\frac{n_3}{n_2}\right)N_2+(1+\mu_1)(1+\mu_2)\left(\frac{n_3 \cdot n_5}{n_2 \cdot n_4}\right)N_3+\cdots\right\};$$

or if we observe that the quantities μ_1, μ_2, μ_3, are composed of terms all of which are of *one* dimension in sin. ϕ, sin. ϕ_1, sin. ϕ_2, &c. and that the quantities N_1, N_2, N_3, &c. (equation 252) are all likewise of one dimension in those exceedingly small quantities; and if we neglect terms above the first dimension in those quantities, then

$$N=\left(\frac{r_1}{a_1}\right)\left\{N_1+\left(\frac{n_3}{n_2}\right)N_2+\left(\frac{n_3 n_5}{n_2 n_4}\right)N_3+\left(\frac{n_3 n_5 n_7}{n_2 n_4 n_6}\right)N_4+\cdots\right\}\ldots(284).$$

If $m=40$, then $p=2{\cdot}88$, with a gain of one third over a single pair.
If $m=3{\cdot}59$, then $p=1$.
If $m=12{\cdot}85$, then $p=2$.
If $m=46{\cdot}3$, then $p=3$.
If $m=166{\cdot}4$, then $p=4$.

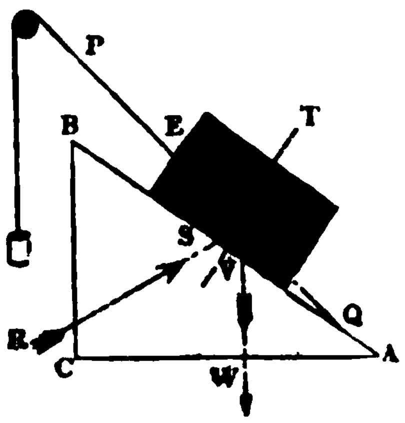
It is evident that when p in any of the above examples appears under the form of a fraction, the nearest whole number to it, must be taken in practice. The influence of the weights of the wheels of the train, and that of the friction of the axes, so greatly however modify these results, that although they are fully sufficient to show the existence in every case of a certain number of wheels, which being assigned to a train destined to produce a given acceleration of motion shall cause that train to produce the required effect with the least expenditure of power, yet they do not in any case determine correctly what that number of wheels should be.

THE INCLINED PLANE.

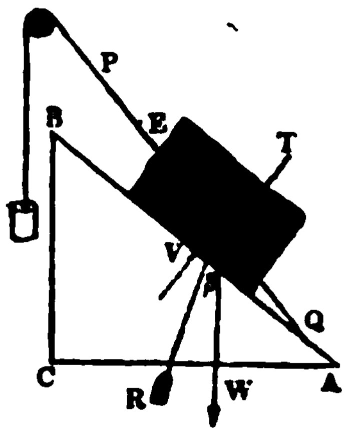
243. Let AB represent the surface of an inclined plane on which is supported a body whose centre of gravity is C, and its weight W, by means of a pressure acting in any direction, and which may be supposed to be supplied by the tension of a cord passing over a pulley and carrying at its extremity a weight.

Let OR represent the direction of the resultant of P and W. If the direction of this line be inclined to the perpendicular ST to the surface of the plane, at an angle OST equal to the limiting angle of resistance, on that side of ST which is farthest from the summit B of the plane (as in *fig.* 1.), the body will be upon the point of slipping *upwards;* and if it be inclined to the perpendicular at an angle OST,

(1.)



(2.)



equal to the limiting angle of resistance, but on the side of ST nearest to the summit B (as in *fig.* 2.), then the body will be upon the point of slipping downwards (Art. 138.); the former condition corresponds to the superior and the latter to the inferior state bordering upon motion (Art. 140.).

Now the resistance of the plane is equal and opposite to the resultant of P and W; let it be represented by R.

There are then three pressures P, W, and R in equilibrium.

$$\therefore \text{(Art. 14.)} \quad \frac{P}{W}=\frac{\sin. WOR}{\sin. POR}.$$

Let $\angle BAC=\iota$, $\angle OST=$ lim^s. $\angle$ of resistance $=\varphi$, let θ

But the four angles of the quadrilateral figure BEOF being equal to four right angles (Euc. 1·32.), $EOF=2\pi-EBF-OEB-OFB$; but $EBF=\iota$, $OEB=\frac{\pi}{2}+\phi_1$, $OFB=\frac{\pi}{2}+\phi_2$. $\therefore EOF=\pi-\iota-\phi_1-\phi_2$.

Similarly, $DOE=2\pi-ADO-AEO-DAE$; but $ADO=\frac{\pi}{2}$, $AEO=\frac{\pi}{2}-\phi_1$, $BAC=\frac{\pi}{2}-\iota$: $\therefore DOE=\frac{\pi}{2}+\iota+\phi_1$.

Since, moreover, DO is parallel to BC, both being perpendicular to AC, $\therefore DOF=\pi-OFC$; but $OFC=\frac{\pi}{2}-\phi_2$: $\therefore DOF=\frac{\pi}{2}+\phi_2$.

$$\therefore \frac{P_1}{R_2}=\frac{\sin.\{\pi-(\iota+\phi_1+\phi_2)\}}{\sin.\left(\frac{\pi}{2}+\phi_2\right)}=\frac{\sin.(\iota+\phi_1+\phi_2)}{\cos.\phi_2};$$

$$\therefore P_1=R_1\frac{\sin.(\iota+\phi_1+\phi_2)}{\cos.\phi_2} \quad \ldots\ldots (299.)$$

$$\frac{P_1}{R_2}=\frac{\sin.\{\pi-(\iota+\phi_1+\phi_2)\}}{\sin.\left(\frac{\pi}{2}+\iota+\phi_1\right)}=\frac{\sin.(\iota+\phi_1+\phi_2)}{\cos.(\iota+\phi_1)};$$

$$\therefore P_1=R_2 . \frac{\sin.(\iota+\phi_1+\phi_2)}{\cos.(\iota+\phi_1)} \quad \ldots\ldots\ldots (300.)$$

In the case in which the surface GH yields to the pressure of the plane, KL remaining fixed, we obtain (equation 121.) for the *modulus* (see Art. 148.), observing that $P_1^{(0)}=R_1 \sin.\iota$ (equation 229),

$$U_1=U_2\frac{\sin.(\iota+\phi_1+\phi_2)}{\sin.\iota . \cos.\phi_2} \quad \ldots\ldots\ldots (301).$$

In the case in which the surface KL yields, CH remaining fixed, observing that $P_1^{(0)}=R_2 \tan.\iota$ (equation 300), we have,

$$U_1=U_2\frac{\sin.(\iota+\phi_1+\phi_2)}{\cos.(\iota+\phi_1)\tan.\iota} \quad \ldots\ldots\ldots (302).$$

cylinder from A to C, and the thread of another, having a different inclination, from D to B.

Let, moreover, the planes A and C (p. 344.) be imagined to be wrapped round two hollow cylindrical surfaces, of equal diameters with the above-mentioned solid cylinder, and contained within the solid pieces E and F, through which hollow cylinders AB passes. Two female screws will thus be generated within the pieces E and F, the helix of the one adapting itself to that of the male screw extending from A to C, and the helix of the other to that upon the male screw extending from D to B. If, then, the piece E be conceived to be fixed, and the piece F moveable in the direction of the length of the screw, but prevented from turning with it by the intervention of a guide, and if a pressure P_1 be applied at A to turn the screw AB, the action of this combination will be precisely analogous to that of the system of inclined planes discussed in Art. 247., and the conditions of the equilibrium precisely the same; so that the relation between the pressure P_1 applied to turn the screw (when estimated at the circumference of the thread) and that P_2, which it may be made to overcome, are determined by equation (306), and its modulus by equation (307).

The invention of the differential screw has been claimed by M. Prony, and by Mr. White of Manchester. A comparatively small pressure may be made by means of it to yield a pressure enormously greater in magnitude.* It admits of numerous applications, and, among the rest, of that suggested in the preceding engraving.

* It will be seen by reference to equation (306), that the working pressure P_2 depends for its amount, not upon the actual inclinations ι_1 ι_2 of the threads, but on the difference of their inclinations; so that its amount may be enormously increased by making the threads nearly of the same inclination. Thus, neglecting friction, we have, by equation (306), $P_2 = P_1 \frac{\cos.\iota_1 \cos.\iota_2}{\sin.(\iota_1-\iota_2)}$; which expression becomes exceedingly great when ι_1 nearly equals ι_2.

$$U=\tfrac{1}{2}\frac{\mu}{g}M\left(1+\frac{K^2}{R^2}\right)V^2 \quad . \; . \; . \; . \quad (374).$$

The accumulated work is therefore the same as though the wheel had moved with a motion of translation only, but with a greater velocity, represented by the expression $\left(1+\frac{K^2}{R^2}\right)^{\frac{1}{2}}V$.

282. On the state of the accelerated or the retarded motion of a machine.

Let the work U_1 done upon the driving point of a machine be conceived to be in excess of that U_2 yielded upon the working points of the machine and that expended upon its prejudicial resistances. Then we have by equation (117)

$$U_1=AU_2+BS_1+\frac{1}{2g}(V^2-V_1^2)\Sigma w\lambda^2;$$

where V represents the velocity of the driving point of the machine after the work U_1 has been done upon it, V_1 that when it began to be done, and $\Sigma w\lambda^2$ the coefficient of equable motion. Now let S_1 represent the space through which U_1 is done, and S_2 that through which U_2 is done; and let the above equation be differentiated in respect to S_1,

$$\therefore \frac{dU_1}{dS_1}=A\frac{dU_2}{dS_2}\cdot\frac{dS_2}{dS_1}+B+\frac{1}{g}V\frac{dV}{dS_1}\Sigma w\lambda^2;$$

but $\frac{dU_1}{dS_1}=P_1$ (Art. 51.), if P_1 represent the driving pressure. Also $\frac{dU_2}{dS_2}=P_2$, if P_2 represent the working pressure; also $V\frac{dV}{dS_1}=V\frac{dV}{dt}\cdot\frac{dt}{dS_1}=V\cdot\frac{dV}{dt}\cdot\frac{1}{V}=\frac{dV}{dt}=f$ (equation 72). If, therefore, we represent by Λ the relation $\frac{dS_2}{dS_1}$, between the spaces described in the same exceedingly small time by the driving and working points, we have

$$P_1=A\Lambda P_2+B+\frac{f}{g}\Sigma w\lambda^2 \quad . \; . \; . \; . \; . \quad (375);$$

and the ratio λ of this space to that described by the driving point of the machine will be represented by

$$\lambda=\left(\frac{b_1 \,.\, b_2 \,.\,.\,.\, b_{p-1}}{a_1 \,.\, a_2 \,.\,.\,.\, a_p}\right)\rho.$$

The sum $\Sigma w\lambda^2$ will therefore be represented in respect to this one element by

$$\left(\frac{b_1 \,.\, b_2 \,.\,.\,.\, b_{p-1}}{a_1 \,.\, a_2 \,.\,.\,.\, a_p}\right)\Sigma w\rho^2.$$

Or if I_p represent the moment of inertia of the element, and μ_p the weight of each cubic unit of its mass, that portion of the value of $\Sigma w\lambda^2$ which depends upon this element will be represented by

$$\left(\frac{b_1 b_2 \,.\,.\,.\,.\, b_{p-1}}{a_1 a_2 \,.\,.\,.\,.\, a_p}\right)^2 I_p \mu_p.$$

And the same being true of every other element of the machine, we have

$$\Sigma w\lambda^2=\Sigma\left(\frac{b_1 b_2 \,.\,.\,.\,.\, b_{p-1}}{a_1 a_2 \,.\,.\,.\,.\, a_p}\right)^2 I_p \mu_p \quad .\;.\;.\;. \quad (377),$$

which is a general expression for the coefficient of equable motion in the case supposed. The value of Λ in equation (366) is evidently represented by

$$\Lambda=\frac{b_1 b_2 b_3 \,.\,.\,.\,.\, b_n}{a_1 a_2 a_3 \,.\,.\,.\,.\, a_n} \quad .\;.\;.\;.\;. \quad (378).$$

284. *To determine the pressure upon the point of contact of any two elements of a machine moving with an accelerated or retarded motion.*

Let p_1 be taken to represent the resistance upon the point of contact of the first element with the second, p_2 that upon the point of contact of the second element of the machine with the third, and so on. Then by equation (375), observing that, P_1 and p_1 representing pressures applied to the same element, $\Sigma w\lambda^2$ is to be taken in this case only in respect to

ture of these two lines — the line of resistance, and the line of pressure: one of these lines, the line of resistance, determining the *point* of application of the resultant of the pressures upon each of the surfaces of contact of the system; and the other, the line of pressure, the *direction* of that resultant.

The determination of both, under their most general forms, lies within the resources of analysis; and general equations for their determination in that case, in which all the surfaces of contact, or joints, are planes — the only case which offers itself as a *practical* case — have been given by the author of this work in the sixth volume of the "Cambridge Philosophical Transactions."

The Stability of a Solid Body.

287. The stability of a solid body may be considered to be greater or less, as a greater or less amount of *work* must be done upon it to overthrow it; or according as the amount of work which must be done upon it to bring it into that position in which it will fall over of its own accord is greater or less. Thus the stability of the solid represented in *fig.* 1. resting on a horizontal plane is greater or less, according as the work which must be done upon it, to bring it into the position represented in *fig.* 2., where its centre of gravity is in the vertical passing through its point of support, is greater or less. Now this work is equal (Art. 60.) to that which would be necessary to raise its whole weight, vertically, through that height by which its centre of gravity is raised, in passing from the one position into the other. Whence it follows that the stability of a solid body resting upon a plane is greater or less, as the product of its weight by the vertical height through which its centre of gravity is

Fig. 1. Fig. 2.

point where the line of resistance intersects the base of the wall, $Cx=m$, $CF=b$, $FEC=\beta$; and let the same notation be taken in other respects as in the preceding articles. Then, since x is a point in the direction of the resultant of the resistances by which the base of the column is sustained, the sum of the moments about that point of the pressure P and half the weight of the shore, supposed to be placed at E *, is equal to the sum of the moments of the thrust Q, and the weight μah of the column; or drawing xM and xN perpendiculars upon the directions of P and Q,

$$P.\overline{xM}+w.\overline{xC}=Q.\overline{xN}+\mu ah.\overline{xK}.$$

Now $xM=\overline{xs}\,\sin.xsM=(HK-Ht)\,\sin.\alpha=\{h-(Hp+st)\cot.\alpha\}\sin.\alpha=h\sin.\alpha-(k+\frac{1}{2}a-m)\cos.\alpha$, $xN=(b+m)\cos.\beta$,

$$\therefore P\{h\sin.\alpha-(k+\tfrac{1}{2}a-m)\cos.\alpha\}+wm=Q(b+m)\cos.\beta+\mu ah(\tfrac{1}{2}a-m).$$

Solving this equation in respect to Q, and reducing, we obtain

$$Q=\frac{P\{h\sin.\alpha-(k+\frac{1}{2}a)\cos.\alpha\}-\frac{1}{2}\mu a^2h+m(P\cos.\alpha+\mu ah+w)}{(b+m)\cos.\beta\,\dagger}\ .\ .\ (386).$$

This expression may be placed under the form

$$\text{'}\cos.\alpha+\mu ah+w)\sec.\beta-\frac{P\{b\cos.\alpha-h\sin.\alpha+(k+\frac{1}{2}a)\cos.\alpha\}+\mu ah(\frac{1}{2}a+b)+wb}{(b+m)\cos.\beta\,\dagger}.$$

If the numerator of the fraction in the second member of this equation be a positive quantity (as in all practical cases it will probably be found to be), the value of Q manifestly diminishes with that of m. Now the least value of m, consistent with the stability of the wall, is zero, since the line of resistance no where intersects the extrados; the least value of Q (the shore being supposed *necessary* to the support of the wall) corresponds, therefore, to the value zero of m; moreover this least value of the thrust upon the shore consistent with

* The weight $2w$ of the shore may be conceived to be divided into two equal parts and collected at its extremities.

† The expression $(b+m)\cos.\beta$ may be placed under the form $b\cot.\beta\sin.\beta+m\cos.\beta=c\sin.\beta+m\cos.\beta$, where c represents the height CE of the point against which the prop rests.

being of a given span L it may be supported with a given degree of stability by walls of a given height h and thickness a; then the same substitutions being made as before, the resulting equation must be solved in respect to ι instead of a.

The value of a admits of a minimum in respect to the variable ι. The value of ι, which determines such a minimum value of a, is that inclination of the rafters which is consistent with the greatest economy in the material of the wall, its stability being given.

307. *The stability of a wall supported by buttresses, and sustaining the pressure of a roof without a tie-beam.*

The conditions of the stability of such a wall, when supported by buttresses of uniform thickness, will evidently be determined, if in equation (393) we substitute for P cos. α and P sin α their values μ_1L sec. ι and $\frac{1}{2}\mu_1$L cosec. ι; we shall thus obtain

$$\mu_1 L(\tfrac{1}{2}h_1 \text{ cosec. } \iota - l \text{ sec. } \iota) = \tfrac{1}{2}\mu(a_1{}^2 h_1 + 2a_1 a_2 h_1 + \frac{1}{n}a_2{}^2 h_2) - m\left\{\mu_1 L \text{ sec. } \iota + \mu(a_1 h_1 + \frac{1}{n}a_2 h_2)\right\} \quad . . (405).$$

From which equation the thickness a_2 of the buttresses necessary to give any required stability m to the wall may be determined.

If the thickness of the buttresses be different at different heights, and they be surmounted by pinnacles, the conditions of the stability are similarly determined by substituting for P sin. α and P cos. α the same values in equations (395) and (397).

To determine the conditions of the stability of a Gothic building, whose nave, having a roof without a tie-beam, is supported by the rafters of its two aisles, or by flying buttresses, which rest upon the summits of the walls of its aisles, a similar substitution must be made in equation (388).

If the walls of the aisles be supported by buttresses, equation (388) must be replaced by a similar relation obtained by the methods lad down in Arts. 301. and 303.; the same substitution for P sin. α and P cos. α must then be made.

$$ON + NI = \tfrac{1}{2}\mu AV(AB + LM) + RN \cos. RNI = \tfrac{1}{2}\mu x\{2a + x(\tan. \alpha_1 - \tan. \alpha_2)\} + P\cos. \iota;$$

$$\therefore \frac{y-\lambda}{x-(\lambda+k)\cot. \iota} = \frac{P \sin. \iota}{\frac{1}{2}\mu x\{2a + x(\tan. \alpha_1 - \tan. \alpha_2)\} + P\cos. \iota}.$$

Transposing and reducing,

$$y = \frac{\frac{1}{2}\lambda\mu x\{2a + x(\tan. \alpha_1 - \tan. \alpha_2)\} + P(x \sin. \iota - k \cos. \iota)}{\frac{1}{2}\mu x\{2a + x(\tan. \alpha_1 - \tan. \alpha_2)\} + P\cos. \iota};$$

but substituting x for c in equation (419), and multiplying both sides of that equation by the denominator of the fraction in the second member, and by the factor $\frac{1}{2}\mu x$, we have

$$\tfrac{1}{2}\lambda\mu x\{2a + x(\tan. \alpha_1 - \tan. \alpha_2)\} = \tfrac{1}{6}\mu x^3(\tan.^2 a - \tan.^2 \alpha_2) + \tfrac{1}{2}\mu x^2 a \tan. \alpha_1 + \tfrac{1}{2}\mu x a^2;$$

$$\therefore y = \frac{\frac{1}{3}\mu x^3(\tan.^2 \alpha_1 - \tan.^2 \alpha_2) + \mu x^2 a \tan. \alpha_1 + \mu x a^2 + 2P(x \sin. \iota - k \cos. \iota)}{\mu x\{2a + x(\tan. \alpha_1 - \tan. \alpha_2)\} + 2P\cos. \iota} \quad . . (420);$$

which is the equation to the line of resistance in a buttress. If the intrados AD be vertical, $\tan. \alpha_2$ is to be assumed $= 0$. If AD be inclined on the opposite side of the vertical to that shown in the figure, $\tan. \alpha_2$ is to be taken negatively. The line of resistance being of three dimensions in x, it follows that, for *certain values* of y, there are three possible values of x; the curve has therefore a point of contrary flexure. The conditions of the equilibrium of the buttress are determined from its line of resistance precisely as those of the wall.

Thus the thickness a of the buttress at its summit being given, and its height c, and it being observed that the distance CE is represented by $a + c \tan. \alpha_1$, the inclination α_1 of its extrados to the vertical may be determined, so that its line of resistance may intersect its foundation at a given distance m from its extrados, by solving equation (420) in respect to $\tan. \alpha_1$, having first substituted c for x and $a + c \tan. \alpha_1 - m$ for y; and any other of the elements determining the conditions of the stability of the buttress may in like manner be determined by solving the equation (the same substitutions being made in it) in respect to that element.

to be *washed* down by the rain), and the surface retains permanently its *natural* slope.

The limiting angle of resistance φ is thus determined by observing what is the natural slope of each description of earth.

The following table contains the results of some such observations* : —

NATURAL SLOPES OF DIFFERENT KINDS OF EARTH.

Nature of Earth.	Natural Slope.	Authority.
Fine dry sand (a single experiment) -	21°	Gadroy.
Ditto - - -	34° 29′	Rondelet.
Ditto - - -	39°	Barlow.
Common earth pulverised and dry -	46° 50′	Rondelet.
Common earth slightly damp -	54°	Rondelet.
Earth the most dense and compact -	55°	Barlow.
Loose shingle perfectly dry -	39°	Pasley.

SPECIFIC GRAVITIES OF DIFFERENT KINDS OF EARTH.

Nature of Earth.	Specific Gravity.
Vegetable earth - - - -	1·4
Sandy earth - - - -	1·6
Marl - - - - -	1·9
Earthy sand - - - -	1·7
Rubble masonry of calcareous earth or siliceous stones -	1·7 to 2·3
Rubble masonry of granite - - -	2·3
Rubble masonry of basaltic stones - -	2·5

321. THE PRESSURE OF EARTH.

Let BD represent the surface of a wall sustaining the pressure of a mass of earth whose surface AE is horizontal.

Let P represent the resultant of the pressures sustained by any portion AX of the wall; and let the cohesion of the

* It is taken from the treatise of M. Navier, entitled *Resumé d'un Cours de Construction*, p. 160.

This maximum value is that which satisfies the conditions

$$\frac{dP}{d\iota}=0, \text{ and } \frac{d^2P}{d\iota^2}<0.$$

But differentiating equation (427) in respect to ι, we obtain by reduction

$$\frac{dP}{d\iota}=\tfrac{1}{4}\mu_1 x^2\frac{\sin.2(\iota+\phi)-\sin.2\iota}{\cos.^2\iota\sin.^2(\iota+\phi)} \quad \ldots\ldots (428).$$

Let the numerator and denominator of the fraction in the second member of this equation be represented respectively by p and q; therefore $\frac{d^2P}{d\iota^2}=\frac{1}{4}\mu_1 x^2\cdot\frac{1}{q^2}\left(\frac{dp}{d\iota}q-\frac{dq}{d\iota}p\right)$; but when $\frac{dP}{d\iota}=0, p=0$; in this case, therefore, $\frac{d^2P}{d\iota^2}=\frac{1}{4}\mu_1 x^2\frac{1}{q}\frac{dp}{d\iota}$. Whence it follows, by substitution, that for every value of ι by which the first condition of a maximum is satisfied, the second differential coefficient becomes

$$\frac{d^2P}{d\iota^2}=\tfrac{1}{2}\mu_1 x^2\frac{\cos.2(\iota+\phi)-\cos.2\iota}{\cos.^2\iota\sin.^2(\iota+\phi)} \quad \ldots\ldots (429).$$

Now it is evident from equation (428) that the condition $\frac{dP}{d\iota}=0$ is satisfied by that value of ι which makes $2(\iota+\phi)=\pi-2\iota$, or

$$\iota=\frac{\pi}{4}-\frac{\phi}{2} \quad \ldots\ldots (430).$$

And if this value be substituted for ι in equation (429), it becomes

$$\frac{d^2P}{d\iota^2}=\mu_1 x^2\frac{-\sin.\phi}{\cos.^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\sin.^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)} \quad \ldots\ldots (431);$$

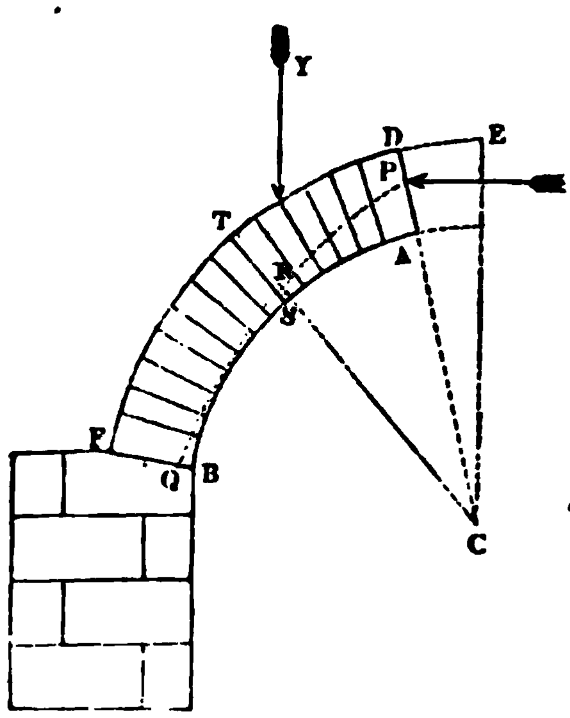
which expression is essentially negative, so that the second condition is also satisfied by this value of ι. It is that, therefore, which corresponds to the maximum value of P; and substituting in equation (427), and reducing, we obtain for this maximum value of P the expression

[redacted]

$$\therefore\ \rho = \frac{Mh + Yx - Xy + Pp}{(M+Y)\sin.\theta + (P-X)\cos.\theta} \quad \ldots\ldots \quad (458),$$

which is the equation to the line of resistance.

M and h are given functions of θ; as also are X and Y, if the pressure of the load extend *continuously* over the surface of the extrados from D to T.



It remains from this equation to determine the pressure P, being that supplied by the opposite semi-arch. As the simplest case, let all the voussoirs of the arch be of the same depth, and let the inclination ECP of the first joint of the semi-arch to the vertical be represented by Θ, and the radii of the extrados and intrados by R and r. Then, by the known principles of statics,

$$Mh = \int_r^R \int_\Theta^\theta r^2 \sin.\theta d\theta dr = -\tfrac{1}{3}(R^3 - r^3)(\cos.\theta - \cos.\Theta);$$

also, $M = \frac{1}{2}(R^2 - r^2)(\theta - \Theta)$;

$$\therefore\ \rho\{\tfrac{1}{2}(R^2 - r^2)(\theta - \Theta)\sin.\theta + Y\sin.\theta - X\cos.\theta + P\cos.\theta\}$$

$$= \tfrac{1}{3}(R^3 - r^3)(\cos.\Theta - \cos.\theta) + Yx - Xy + Pp \quad \ldots\ldots \quad (459),$$

which is the general equation to the line of resistance.

The Angle of Rupture.

339. At the points of rupture the line of resistance *meets* the intrados, so that there $\rho = r$: if then Ψ be the corresponding value of θ,

$$r\{\tfrac{1}{2}(R^2 - r^2)(\Psi - \Theta)\sin.\Psi + Y\sin.\Psi - X\cos.\Psi + P\cos.\Psi\}$$

$$= \tfrac{1}{3}(R^3 - r^3)(\cos.\Theta - \cos.\Psi) + Yx - Xy + Pp \quad \ldots\ldots \quad (460).$$

Also at the points of rupture the line of resistance *touches*

the intrados, so that there $\frac{d\rho}{d\theta}=\frac{dr}{d\theta}=0$; assuming then, to simplify the results, that the pressure of the load is wholly in a vertical direction, so that $X=0$, and that it is collected over a single point of the extrados, so that $\frac{dY}{d\theta}=0$, and differentiating equation (459), and assuming $\frac{d\rho}{d\theta}=0$, when $\theta=\Psi$ and $\rho=r$, we obtain

$$r\left\{\tfrac{1}{2}(R^2-r^2)(\Psi-\Theta)\cos.\Psi+\tfrac{1}{2}(R^2-r^2)\sin.\Psi+Y\cos.\Psi-P\sin.\Psi\right\}=\tfrac{1}{3}(R^3-r^3)\sin.\Psi$$

hence, assuming $R=r(1+\alpha)$,

$$\left\{\frac{6P}{r^3}+a^2(2a+3)\right\}\tan.\Psi=\left\{\frac{6Y}{r^2}-3a(a+2)\Theta\right\}+3a(a+2)\Psi\;\ldots\;(461).$$

Eliminating $(\Psi-\Theta)$ between equations (460) and (461), we have

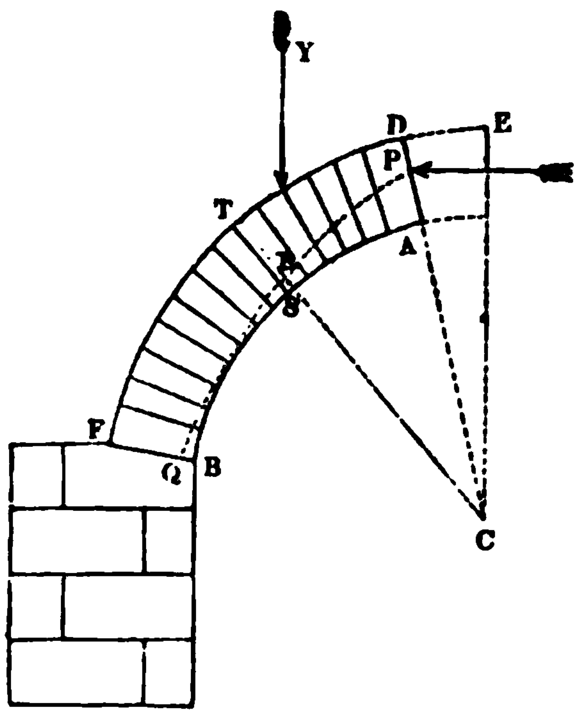
$$\left\{\frac{P}{r^3}+a^2(\tfrac{1}{3}a+\tfrac{1}{2})\right\}\sec.^2\Psi-\left\{\frac{Yx+Pp}{r^3}+a(\tfrac{1}{3}a^2+a+1)\cos.\Theta\right\}\sec.\Psi=-a(\tfrac{1}{2}a+1)\;\ldots\;(462).$$

Eliminating P between equations (460) and (461), and reducing,

$$\frac{Y^*}{r^2}\left\{\frac{p\cos.\Psi+x\sin.\Psi}{r}-1\right\}=(\tfrac{1}{2}a^2+a)\left(1-\frac{p}{r}\cos.\Psi\right)(\Psi-\Theta)+\frac{p}{r}(\tfrac{1}{2}a^2+\tfrac{1}{3}a^3)\sin.\Psi$$

$$-\left\{(a+a^2+\tfrac{1}{3}a^3)\cos.\Theta-(\tfrac{1}{2}a^2+a)\cos.\Psi\right\}\sin.\Psi\;\ldots\;(463).$$

* This equation might have been obtained by differentiating equation (459) in respect to P and θ, and assuming $\frac{dP}{d\theta}=0$ when r and Ψ are substituted for ρ and θ; for if that equation be represented by $u=0$, u being a function of P, ρ and θ, $\frac{du}{dP}\frac{dP}{d\theta}+\frac{du}{d\theta}=0$, and $\frac{du}{d\rho}\frac{d\rho}{d\theta}+\frac{du}{d\theta}=0$. The same result $\frac{du}{d\theta}=0$ is therefore obtained, whether we assume $\frac{dP}{d\theta}=0$, or $\frac{d\rho}{d\theta}=0$, which last supposition is that made in equation (461), whence equation (463) has resulted. The hypotheses $\frac{dP}{d\theta}=0$, $\rho=r$, determine the minimum of the pressures P, which being applied to a given point of the key-stone will prevent the semi-arch from turning on any of the successive joints of its voussoirs.



Let $AP = \lambda r$; therefore $\frac{p}{r} = (1+\lambda)\cos.\Theta$. Substituting this value of $\frac{p}{r}$,

$$\left.\frac{x}{r}\sin.\Psi + (1+\lambda)\cos.\Theta\cos.\Psi - 1\right\} = (\tfrac{1}{2}a^2 + a)\left\{\{1-(1+\lambda)\cos.\Theta\cos.\Psi\}(\Psi-\Theta)\right.$$

$$\left. + (\cos.\Psi - \cos.\Theta)\sin.\Psi\right\} + \lambda(\tfrac{1}{2}a^2 + \tfrac{1}{3}a^3)\sin.\Psi\cos.\Theta \;.\;.\;.\;.\;.\; (464),$$

by which equation the angle of rupture Ψ is determined.

If the arch be a continuous segment the joint AD is vertically above the centre, and CD coinciding with CE, $\Theta = 0$; if it be a broken segment, as in the Gothic arch, Θ has a given value determined by the character of the arch. In the pure or equilateral Gothic arch, $\Theta = 30°$. Assuming $\Theta = 0$, and reducing,

$$\frac{Y}{r^2}\left\{\frac{x}{r} - \left(\tan.\frac{\Psi}{2} - \lambda\cot.\Psi\right)\right\} = (\tfrac{1}{2}a^2 + a)\left\{\left(\tan.\frac{\Psi}{2} - \lambda\cot.\Psi\right)\Psi - \text{vers.}\Psi\right\} + \lambda(\tfrac{1}{2}a^2 + \tfrac{1}{3}a^3) \ldots (465).$$

It may easily be shown that as Ψ increases in this equation Y increases, and conversely; so that as the load is increased the points of rupture descend. When $Y = 0$, or there is no load upon the extrados,

$$\left(\tan.\frac{\Psi}{2} - \lambda\cot.\Psi\right)\Psi - \text{vers.}\Psi + \tfrac{1}{3}\lambda a\frac{3+2a}{2+a} = 0 \;.\;.\;.\;.\;.\; (466).$$

When $x = 0$, or the load is placed on the crown of the arch,

$$\frac{Y}{r^2} = \frac{(\frac{1}{2}a^2 + a)\text{vers.}\Psi - \lambda(\frac{1}{2}a^2 + \frac{1}{3}a^3)}{\tan.\frac{\Psi}{2} - \lambda\cot.\Psi} - (\tfrac{1}{2}a^2 + a)\Psi \;.\;.\;.\;.\;.\; (467).$$

[redacted]

$$\therefore\ TV=R\{1+\beta-\cos.(\theta-\iota)\sec.\iota\};$$

also, $x=\overline{DZ}=R\sin.\theta$;

$$\therefore \text{ area GFTV}=\int_{\Theta}^{\theta} TV\cdot\frac{dx}{d\theta}d\theta=R^2\int_{\Theta}^{\theta}\{1+\beta-\cos.(\theta-\iota)\sec.\iota\}\cos.\theta d\theta;$$

$$\therefore Y=\text{weight of mass GFTV}=\mu R^3\int_{\Theta}^{\theta}\{1+\beta-\sec.\iota\cos.(\theta-\iota)\}\cos.\theta d\theta=$$

$$^3\Big\{(1+\beta)(\sin.\theta-\sin.\Theta)-\tfrac{1}{4}\sec.\iota\{\sin.(2\theta-\iota)-\sin.(2\Theta-\iota)\}-\tfrac{1}{2}(\theta-\Theta)\Big\}\ldots(469).$$

$$Yx=\text{momentum of GFTV}=\mu R^3\int_{\Theta}^{\theta}\{(1+\beta)-\sec.\iota\cos.(\theta-\iota)\}\sin.\theta\cos.\theta d\theta=$$

$$^3\{\tfrac{1}{2}(1+\beta)(\cos.^2\Theta-\cos.^2\theta)-\tfrac{1}{3}(\cos.^3\Theta-\cos.^3\theta)-\tfrac{1}{3}\tan.\iota(\sin.^3\theta-\sin.^3\Theta)\}\ldots(470).$$

A SEGMENTAL ARCH WHOSE EXTRADOS IS HORIZONTAL.

341. As the simplest case, let us first suppose DV horizontal, the material of the loading similar to that of the arch, and the crown of the arch at A, so that $\iota=0$, $\mu=1$, and $\Theta=0$. Substituting the values of Y and Yx (equations 469, 470) which result from these suppositions, in equation (460), solving that equation in respect to $\frac{P}{r^2}$, and reducing, we have,

$$\frac{P}{r^2}=\frac{\tfrac{1}{3}(1-\alpha)(1+\alpha)^3(1+\beta)\sin.^3\Psi+\tfrac{1}{6}(1+\alpha)^3(1-2\alpha)\cos.^3\Psi+(\tfrac{1}{2}\alpha^2+\tfrac{1}{3}\alpha^3-\tfrac{1}{6})\cos.\Psi-\tfrac{1}{2}\Psi\sin.\Psi+\tfrac{1}{6}}{1+\lambda-\cos.\Psi}\ldots(471).$$

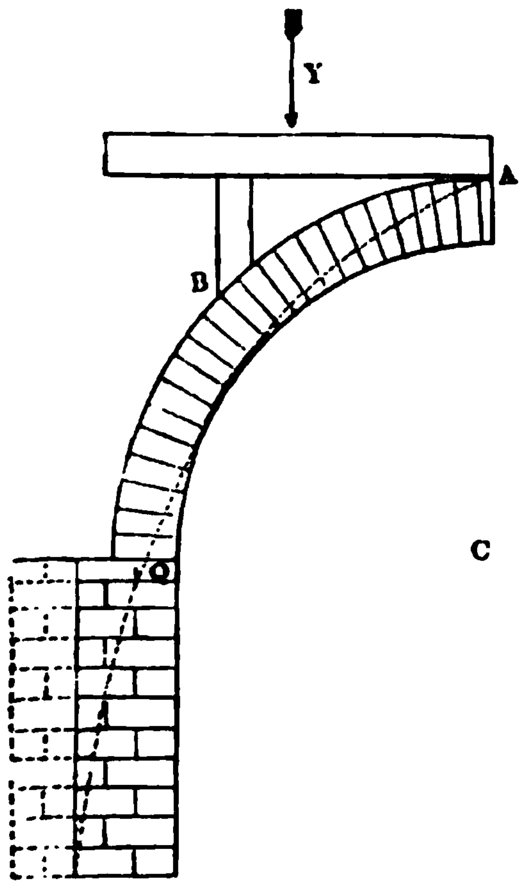
Assuming $\frac{dP}{d\Psi}=0$ (see note, page 468.), and $\lambda=\alpha$, and reducing,

$$\tfrac{1}{3}(1-2\alpha)\cos.^3\Psi-\Big\{(1-\alpha)(1+\beta)+(1+\alpha)(1-2\alpha)\Big\}\cos.^2\Psi+\Big\{\frac{1}{(1+\alpha)^3}+2(1-\alpha^2)(1+\beta)\Big\}\cos.\Psi$$
$$+\frac{1}{(1+\alpha)^3}\{1-(1+\alpha)\cos.\Psi\}\frac{\Psi}{\sin.\Psi}-(1-\alpha)(1+\beta)-\frac{2}{3(1+\alpha)^3}-\frac{1+\frac{1}{3}\alpha^3}{1+\alpha}=0\ldots(472).$$

In the case in which the line of resistance passes through

Then assigning one half of the load upon the crown to each semi-arch, and calling x the horizontal distance of the centre of gravity of the load upon either semi-arch from C, it may easily be calculated that $\frac{x}{r}=\frac{3}{4}\sin.45=\cdot5303301$. Hence it appears from equation (468) that no loading can cause the angle of rupture to exceed 65°. Assume it to equal 60°; the amount of the load necessary to produce this angle of rupture, when distributed as above, will then be determined by assuming in equation (465), $\Psi=60°$, and substituting α for λ, $\cdot2$ for α, and $\cdot5303301$ for $\frac{x}{r}$. We thus obtain $\frac{Y}{r^2}=\cdot0138$. Substituting this value of $\frac{Y}{r^2}$, and also the given values of α and Ψ in equation (462), and observing that in this equation $\frac{p}{r}$ is to be taken $=1+\alpha$ and $\Theta=0$, we find $\frac{P}{r^2}=\cdot11832$. Substituting this value of $\frac{P}{r^2}$ in the equation (459), we have for the final equation to the line of resistance beneath the point B

$$\rho=r\cdot\frac{\cdot2426\,\text{vers.}\,\theta+\cdot1493}{\cdot0138\sin.\theta+\cdot1183\cos.\theta+\cdot22\,\theta\sin.\theta}.$$



If the arc of the arch be a complete semicircle, the value of ρ in this equation corresponding to $\theta=\frac{\pi}{2}$ will determine the point Q, where the line of resistance intersects the abutment; this value is $\rho=1\cdot09\,r$.

If the arc of the arch be the third of a circle, the value of ρ at the abutment is that corresponding to $\theta=\frac{\pi}{3}$; this will be found to be r, as it manifestly ought to be, since the points of rupture are in this case at the springing.

the distance of the point R from its centre of gravity, then (Art. 18.)

$$kh=\Sigma_0^{h_1}\rho\Delta k-\Sigma_0^{h_2}\rho\Delta k; \quad \therefore \ P_1 \sin.\theta=\frac{Ekh}{R};$$

$$\therefore \ h=\frac{RP_1}{Ek}\sin.\theta \ . \ . \ . \ . \ (504);$$

which expression represents the distance of the neutral line from the centre of gravity of any section PT of the lamina, that distance being measured towards the extended or the compressed side of the lamina according as θ is positive or negative; so that the neutral line passes from one side to the other of the line joining the centres of gravity of the cross sections of the lamina, at the point where $\theta=0$, or at the point where the normal to the neutral line is parallel to the direction of P_1.

358. *Case of a rectangular beam.*

If the form of the beam be such that it may be divided into laminæ parallel to ABCD of similar forms and equal dimensions, and if the pressure P_1 applied to each lamina may be conceived to be the same; or if its section be a rectangle, and the pressures applied to it be applied (as they usually are) uniformly across its width, then will the distance h of the neutral line of each lamina from the centre of gravity of any cross section of that lamina, such as PT, be the same, in respect to corresponding points of all the laminæ, whatever may be the deflexion of the beam; so that in this case the neutral surface is always a cylindrical surface.

359. *Case in which the deflecting pressure* P_1 *is nearly perpendicular to the length of the beam.*

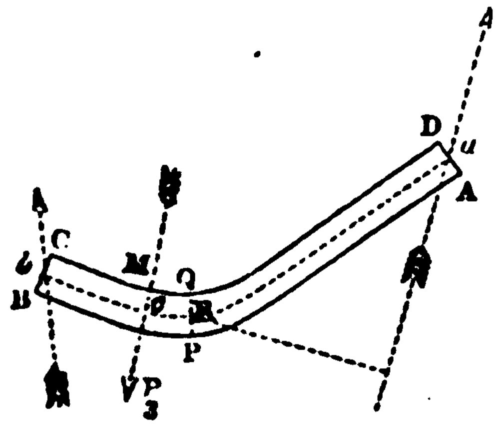
In this case θ, and therefore $\sin.\theta$, is exceedingly small, so long as the deflexion is small at every point R of the neutral

$\frac{P_1^2 p_1^2}{E^2 I^2}$; whence, by substitution, the above expression becomes $\frac{1}{2}\frac{P_1^2}{E}\sum_0^1 \frac{p_1^2}{I}\Delta x$. Passing to the limit, and representing the work expended upon the deflexion of the part AM of the beam by u_1,

$$u_1 = \frac{P_1^2}{2E}\int_0^{a_1} \frac{p_1^2}{I} dx \quad \dots\dots \quad (510).$$

369. *The work expended upon the deflexion of a beam of uniform dimensions, when the deflecting pressures are nearly perpendicular to the surface of the beam.*

In this case I is constant, and $p_1 = x$; whence we obtain by integrating (equation 510) between the limits 0 and a_1,



$$u_1 = \frac{1}{6}\frac{P_1^2 a_1^3}{EI} \quad \dots\dots \quad (511).$$

where u_1 represents the work expended upon the deflexion of the portion AM of the beam. Similarly, if $bc = a_2$, the work expended upon the deflexion of the portion BM of the beam is represented by

$$u_2 = \frac{1}{6}\frac{P_2^2 a_2^3}{EI};$$

so that the whole work U_3 expended upon the deflexion of the beam is represented by

$$U_3 = \frac{P_1^2 a_1^3 + P_2^2 a_2^3}{6EI}.$$

But by the principle of the equality of moments, if a represent the whole length of the beam,

$$P_1 a = P_3 a_2, \quad P_2 a = P_3 a_1.$$

Eliminating P_1 and P_2 between these equations and the preceding, we obtain by reduction

perpendicular to the surface of the beam at the point of application of P_3 by D_3, we shall obtain

$$D_3 = \frac{(a_1 a_2)^2 P_3}{3EIa} \text{*} \quad \ldots . \quad (515).$$

If the pressure P_3 be applied at the centre of the beam $a_1 = a_2 = \frac{1}{2}a$,

$$\therefore \; D_3 = \frac{a^3 P_3}{48EI} \quad \ldots . \quad (516).$$

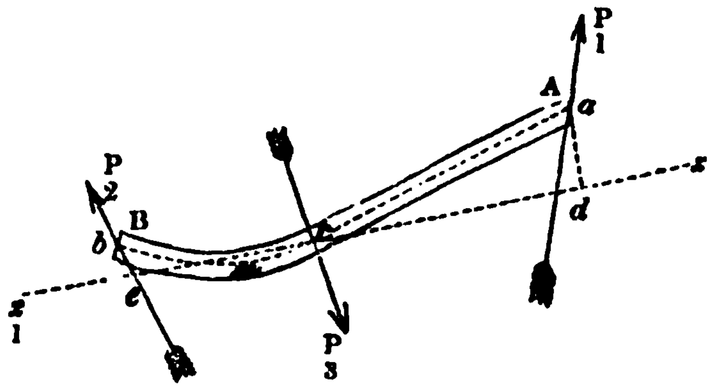
Eliminating P_1 between equations (511) and (514), and P_3 between equations (512) and (515), we obtain

$$u_1 = \frac{3EID_1^2}{2a_1^3}, \quad U_3 = \frac{3aEID_3^2}{2(a_1 a_2)^2} \quad \ldots . \quad (517);$$

by which equations the work expended upon the deflexion of a beam is determined in terms of the *deflexion* itself, as by equations (511) and (512) it was determined in terms of the deflecting *pressures*.

371. Conditions of the deflexion of a beam to which are applied three pressures, whose directions are nearly perpendicular to its surface.

Let AB represent any lamina of the beam parallel to its



plane of deflexion, and *acb* the neutral line of that lamina intersected by the direction of P_3 in the point *c*.

Draw xx_1 parallel to the length of the beam before its deflexion, and take this line as the axis of the abscissæ, and the point *c* as the origin; then, representing by x and y the co-

* This result is identical with that obtained by a different method of investigation by M. Navier (*Resumè de Leçons de Construction*, Art. 359.).

$$\frac{dy}{dx}+\tan.\beta=\frac{P_2}{EI}\{a_2x-\tfrac{1}{2}x^2\} \quad . \; . \; . \; . \quad (521).$$

$$y=\frac{P_2}{EI}\{\tfrac{1}{2}a_2x^2-\tfrac{1}{6}x^3\}-x\tan.\beta \quad . \; . \; . \; . \quad (522).$$

If D_1 and D_2 be taken to represent the deflexions at the points a and b, and ca and cb be assumed respectively equal to cd and ce,

$$\text{by equation (520), } D_1=\frac{P_1a_1^3}{3EI}+a_1\tan.\beta,$$

$$\text{by equation (522), } D_2=\frac{P_2a_2^3}{3EI}-a_2\tan.\beta.$$

If the pressures P_1 and P_2 be supplied by the resistances of fixed surfaces, then $D_1=D_2$. Subtracting the above equation we obtain, on this supposition,

$$0=\frac{P_1a_1^3-P_2a_2^3}{3EI}+(a_1+a_2)\tan.\beta.$$

Now $P_1a_1^3-P_2a_2^3=\dfrac{P_3a_2a_1^3-P_3a_1a_2^3}{a}=P_3a_1a_2(a_1-a_2)$; observing that $P_1a=P_3a_2$, $P_2a=P_3a_1$, and $a_1+a_2=a$,

$$\therefore\ \tan.\beta=\frac{P_3a_1a_2(a_2-a_1)}{3EIa} \quad . \; . \; . \; . \quad (523).$$

If β_1, β_2 represent the inclinations of the neutral line to xx_1 at the points a and b, then by equations (519) and (521)

$$\tan.\beta_1-\tan.\beta=\frac{P_1a_1^2}{2EI},\ \tan.\beta_2+\tan.\beta=\frac{P_2a_2^2}{2EI}.$$

Substituting for $\tan.\beta$ its value from equation (523), eliminating and reducing,

$$\tan.\beta_1=\frac{P_3a_1a_2(a_1+2a_2)}{6EIa},\ \tan.\beta=\frac{P_3a_1a_2(a_2+2a_1)}{6EIa} \quad . \; . \; . \quad (524).$$

To determine the point m where the tangent to the neutral line is parallel to cxx_1, or to the undeflected position of the beam, we must assume $\dfrac{dy}{dx}=0$ in equatio n(521); if we then substitute for $\tan.\beta$ its value from equation (523), substitute for P_2 its value in terms of P_3, and solve the re-

[redacted]

$$\therefore \ s=a+\frac{P^2a^5}{60E^2I^2} \ \ldots \ (526).$$

Eliminating P between this equation and equation (516), and representing the deflexion by D,

$$s=a+\tfrac{3}{5}\frac{D^2}{a} \ *.$$

373. A BEAM, ONE PORTION OF WHICH IS FIRMLY INSERTED IN MASONRY, AND WHICH SUSTAINS A LOAD UNIFORMLY DISTRIBUTED OVER ITS REMAINING PORTION.

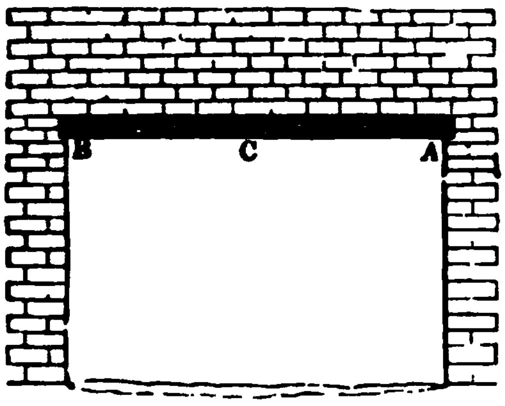
Let the co-ordinates of the neutral line be measured from the point B where the beam is inserted in the masonry, and let the length of the portion AD which sustains the load be represented by a, and the load upon each unit of its length by μ; then, representing by x and y the co-ordinates of any point P of the neutral line,

* The following experiments were made by Mr. Hatcher, superintendant of the work-shop at King's College, to verify this result, which is identical with that obtained by M. Navier (*Resumé de Leçons*, Art. 86.). Wrought iron rollers ·7 inch in diameter were placed loosely on wrought iron bars, the surfaces of contact being smoothed with the file and well oiled. The bar to be tested had a square section, whose side was ·7 inch, and was supported on the two rollers, which were adjusted to 10 feet apart (centre to centre) when the deflecting weight had been put on the bar. On removing the weights carefully, the distance to which the rollers receded as the bar recovered its horizontal position was noted.

Deflecting Weight in lbs.	Deflection in Inches.	Distance through which each Roller receded in Inches.	Distance through which each Roller should have receded by Formula.
56	3·7	·1	·13
84	5·45	·2	·29

374. A BEAM SUPPORTED AT ITS EXTREMITIES, AND SUSTAINING A LOAD UNIFORMLY DISTRIBUTED OVER ITS LENGTH.

Let the length of the beam be represented by $2a$, the load upon each unit of length by μ; take x and y as the co-ordinates of any point P of the neutral line, from the origin A; and let it be observed that the forces applied to AP, and in equilibrium, are the load μx upon that portion of the beam, which may be supposed collected over its middle point, the resistance upon the point A, which is represented by μa, and the elastic forces developed upon the section at P; then by Art. 362.,



$$EI\frac{d^2y}{dx^2}=\tfrac{1}{2}\mu x^2-\mu ax \quad . \ . \ . \ . \quad (532).$$

Integrating this equation between the limits x and a, and observing that at the latter limit $\frac{dy}{dx}=0$, since y evidently attains its maximum value at the middle C of the beam,

$$EI\frac{dy}{dx}=\tfrac{1}{6}\mu(x^3-a^3)-\tfrac{1}{2}\mu a(x^2-a^2) \quad . \ . \ . \ . \quad (533).$$

Integrating a second time between the limits 0 and x, and observing that when $x=0$, $y=0$,

$$EIy=\tfrac{1}{6}\mu(\tfrac{1}{4}x^4-a^3x)-\tfrac{1}{2}\mu a(\tfrac{1}{3}x^3-a^2x) \quad . \ . \ . \ . \quad (534),$$

which is the equation to the neutral line. Substituting a for x in this equation, and observing that the corresponding value of y represents the deflexion D in the centre of the beam, we have by reduction

$$D=\frac{5\mu a^4}{24EI} \quad . \ . \ . \ . \quad (535).$$

Representing by β the inclination to the horizon of the tan-

Since, moreover, the forces impressed upon any portion CQ of the beam, terminating between A and E, are the elastic forces developed upon the transverse section at Q, the resistance μa of the support at A, and the load upon CQ, whose moment about Q is represented by $\frac{1}{2}\mu\overline{CQ}^2$, we have (equation 506), representing CQ by x,

$$EI\frac{d^2y}{dx^2}=\tfrac{1}{2}\mu x^2-\mu a(x-a_1) \;.\;.\;.\;.\; (538).$$

Representing the inclination to the horizon of the tangent to the neutral line at A by β, dividing equation (537) by μ, integrating it between the limits x and a_1, and observing that at the latter limit $\frac{dy}{dx}=\tan.\beta$, we have, in respect to the portion CA of the beam,

$$\frac{EI}{\mu}\left(\frac{dy}{dx}-\tan.\beta\right)=\tfrac{1}{6}x^3-\tfrac{1}{6}a_1^3 \;.\;.\;.\;.\; (539).$$

Integrating equation (538) between the limits x and a, and observing that at the latter limit $\frac{dy}{dx}=0$, since the neutral line at E is parallel to the horizon,

$$\frac{EI}{\mu}\frac{dy}{dx}=\tfrac{1}{6}x^3-\tfrac{1}{2}a(x-a_1)^2-\tfrac{1}{6}a^3+\tfrac{1}{2}a(a-a_1)^2 \;.\;.\;.\;.\; (540);$$

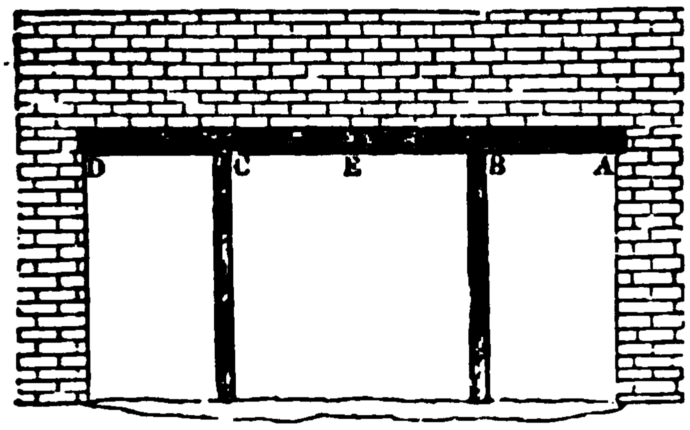
which equation having reference to the portion AE of the beam, it is evident that when $x=a_1$, $\frac{dy}{dx}=\tan.\beta$.

$$\therefore \frac{EI}{\mu}\tan.\beta=\tfrac{1}{2}a(a-a_1)^2-\tfrac{1}{6}(a^3-a_1^3)=\tfrac{1}{6}(a-a_1)(2a^2-4aa_1-a_1^2) \;.\;.\;. (541).$$

Substituting, therefore, for $\tan.\beta$ in equation (539), and reducing, that equation becomes

$$\frac{EI}{\mu}\frac{dy}{dx}=\tfrac{1}{6}x^3+\tfrac{1}{2}a(a-a_1)^2-\tfrac{1}{6}a^3 \;.\;.\;.\;.\; (542).$$

Integrating equation (540) between the limits a_1 and x, and equation (542) between the limits 0 and x, and representing the deflexion at C, and therefore the value of y at A, by D_1,



pressure upon that point be represented by P_1, and the pressure upon B by P_2; also the load upon each unit of the length of the beam by μ.

If P be any point in the neutral line to the portion AB of the beam, whose co-ordinates are x and y, the pressures applied to AP, and in equilibrium, are the pressure P_1 at A, the load μx supported by AP, and producing the same effect as though it were collected over the centre of that portion of the beam, and the elastic forces developed upon the transverse section of the beam at P; whence it follows (Art. 362.) by the principle of the equality of moments, taking P as the point from which the moments are measured, that

$$\mathrm{EI}\frac{d^2y}{dx^2}=\tfrac{1}{2}\mu x^2-\mathrm{P}_1 x \quad . \quad . \quad . \quad . \quad (548).$$

Integrating this equation between the limits a_1 and x, and representing the inclination to the horizon of the tangent to the neutral line at B by β_2,

$$\mathrm{EI}\left(\frac{dy}{dx}-\tan.\beta_2\right)=\tfrac{1}{6}\mu(x^3-a_1^3)-\tfrac{1}{2}\mathrm{P}_1(x^2-a_1^2) \quad . \quad . \quad . \quad . \quad (549).$$

Integrating again between the limits 0 and x,

$$\mathrm{EI}(y-x\tan.\beta_2)=\tfrac{1}{6}\mu(\tfrac{1}{4}x^4-a_1^3x)-\tfrac{1}{2}\mathrm{P}_1(\tfrac{1}{3}x^3-a_1^2x) \ . \ . \ . \ (550).$$

Whence observing that when $x=a_1$, $y=0$,

$$\mathrm{EI}\tan.\beta_2=\tfrac{1}{8}\mu a_1^3-\tfrac{1}{3}\mathrm{P}_1 a_1^2 \quad . \quad . \quad . \quad . \quad . \quad (551).$$

Similarly observing, that if x and y be taken to represent the co-ordinates of a point Q in the beam between B and C, the pressures applied to AQ are the elastic forces upon the section at Q, the pressures P_1 and P_2 and the load μx, we have

$$\mathrm{EI}\frac{d^2y}{dx^2}=\tfrac{1}{2}\mu x^2-\mathrm{P}_1x-\mathrm{P}_2(x-a_1) \quad . \quad . \quad . \quad . \quad (552).$$

Integrating this equation between the limits a_1 and x, and

the same analysis which determines the conditions of the equilibrium in this case will be found applicable in the more general case. Let P_1, P_3, P_5 be taken to represent the resistances of the several points of support, a_1 and a_2 the distances between them, P_2 P_4 the deflecting pressures, and x, y the co-ordinates of any point in the neutral line from the origin B. Substituting in equation (505) for $\frac{1}{R}$ its value $\frac{d^2y}{dx^2}$, and observing that in respect to the portion BD of the beam $\Sigma Pp = P_2(\frac{1}{2}a_1 - x) - P_1(a_1 - x)$, and that in respect to the portion DA of the beam, $\Sigma Pp = -P_1(a_1 - x)$, we have for the differential equation to the neutral line between B and D

$$EI\frac{d^2y}{dx^2} = P_2(\tfrac{1}{2}a_1 - x) - P_1(a_1 - x) \quad . \; . \; . \; . \quad (568),$$

between D and A

$$EI\frac{d^2y}{dx^2} = -P_1(a_1 - x) \quad . \; . \; . \; . \quad (569).$$

Representing by β the inclination of the tangent at B to the axis of the abscissæ, and integrating the former of these equations twice between the limits 0 and x,

$$EI\frac{dy}{dx} = \tfrac{1}{2}P_2(a_1x - x^2) - P_1(a_1x - \tfrac{1}{2}x^2) + EI\tan.\beta \quad . \; . \; . \quad (570);$$

$$EIy = \tfrac{1}{2}P_2(\tfrac{1}{2}a_1x^2 - \tfrac{1}{3}x^3) - \tfrac{1}{2}P_1(a_1x^2 - \tfrac{1}{3}x^3) + EIx\tan.\beta \ldots (571).$$

Substituting $\frac{1}{2}a_1$ for x in these equations, and representing by D_1 the value of y, and by γ the inclination to the horizon of the tangent at the point D, we obtain

$$EI\tan.\gamma = \tfrac{1}{8}P_2a_1^2 - \tfrac{3}{8}P_1a_1^2 + EI\tan.\beta \quad . \; . \; . \; . \quad (572),$$

$$EID_1 = \tfrac{1}{24}P_2a_1^3 - \tfrac{5}{48}P_1a_1^3 + \tfrac{1}{2}EIa_1\tan.\beta \quad . \; . \; . \; . \quad (573).$$

Integrating equation (569) between the limits $\frac{a_1}{2}$ and x,

$$EI\frac{dy}{dx} = -P_1(a_1x - \tfrac{1}{2}x^2) + EI\tan.\gamma + \tfrac{3}{8}P_1a^2.$$

Eliminating tan. γ between this equation and equation (572) and reducing,

$$EI\frac{dy}{dx}=-P_1(a_1x-\tfrac{1}{2}x^2)+EI\tan.\beta+\tfrac{1}{8}P_2a_1^2 \quad \ldots\ldots \quad (574).$$

Integrating again between the limits $\frac{a}{2}$ and x, and eliminating the value of D_1 from equation (573),

$$EIy=-\tfrac{1}{2}P_1(a_1x^2-\tfrac{1}{3}x^3)+(EI\tan.\beta+\tfrac{1}{8}P_2a_1^2)x-\tfrac{1}{48}P_2a_1^3 \quad \ldots\ldots \quad (575).$$

Now it is evident that the equation to the neutral line in respect to the portion CE of the beam, will be determined by writing in the above equation P_5 and P_4 for P_1 and P_2 respectively.

Making this substitution in equation (575), and writing $-\tan.\beta$ for $+\tan.\beta$ in the resulting equation; then assuming $x=a_1$ in equation (575), and $x=a_2$ in the equation thus derived from it, and observing that y then becomes zero in both, we obtain

$$0=-\tfrac{1}{3}P_1a_1^3+\tfrac{5}{48}P_2a_1^3+EIa_1\tan.\beta,$$

$$0=-\tfrac{1}{3}P_5a_2^3+\tfrac{5}{48}P_4a_2^3-EIa_2\tan.\beta.$$

Also, by the general conditions of the equilibrium of parallel pressures (Art. 15.),

$$P_1a_1+\tfrac{1}{2}P_4a_2=P_5a_2+\tfrac{1}{2}P_2a_1,$$

$$P_1+P_3+P_5=P_2+P_4.$$

Eliminating between these equations and the preceding, assuming $a_1+a_2=a$, and reducing, we obtain

$$P_1=\frac{P_2a_1(8a_2+5a_1)-3P_4a_2^2}{16aa_1} \quad \ldots\ldots \quad (576).$$

$$P_5=\frac{P_4a_2(8a_1+5a_2)-3P_2a_1^2}{16aa_2} \quad \ldots\ldots \quad (577).$$

$$P_3=\tfrac{1}{2}\left\{P_2\left(1+\frac{3a_1}{8a_2}\right)+P_4\left(1+\frac{3a_2}{8a_1}\right)\right\} \quad \ldots\ldots \quad (578).$$

By equation (573),

$$D_1=\frac{a_1^2}{768EIa}\left\{P_2a_1(16a_2+7a_1)-9P_4a_2^2\right\}\quad\ldots\ldots(579).$$

Similarly,

$$D_2=\frac{a_2^2}{768EIa}\left\{P_4a_2(16a_1+7a_2)-9P_2a_1^2\right\}\quad\ldots\ldots(580);$$

$$\tan.\beta=\frac{a_1}{48EIa}\left\{P_2(8a_2^2-5a_1^2)-3P_4a_2^2\right\}\quad\ldots\ldots(581).$$

By equation (572),

$$\tan.\gamma=\frac{a_1}{128EIa}\left\{P_2a_1^2+P_4a_2^2\right\}\quad\ldots\ldots(582).$$

If a_1 be substituted for x in equation (574), and for P_1 and $\tan.\beta$ their values from equations (576) and (581); and if the inclination of the tangent at A to the axis of x be represented by β_1, we shall obtain by reduction

$$\tan.\beta_1=\frac{a_1}{32EIa}\left\{P_4a_2^2-P_2a_1(2a_2+a_1)\right\},\quad\ldots\ldots(583).$$

Similarly, if β_2 represent the inclination of the tangent at C to the axis of x,

$$\tan.\beta_2=\frac{a_2}{32EIa}\left\{P_2a_1^2-P_4a_2(2a_1+a_2)\right\}\quad\ldots\ldots(584).$$

380. If the pressures P_2 and P_4, and also the distances a_1 and a_2, be equal,

$$P_1=P_5=\tfrac{5}{16}P_2,\ P_3=\tfrac{11}{8}P_2,\ \tan.\beta=0,\ \tan.\beta_1=\tan.\beta_2=-\frac{P_2a_1^2}{32EI}.$$

381. If the distances a_1 and a_2 be equal, and $P_4=3P_2$,

$$P_1=\tfrac{1}{8}P_2,\ P_3=\tfrac{11}{4}P_2,\ P_5=\tfrac{9}{8}P_2,\ \tan.\beta=-\frac{P_2a_1^2}{16EI},\ \tan.\beta_1=0.^*$$

* The following experiments were made by Mr. Hatcher to verify this result. The bar ACB, on which the experiment was to be tried, was supported on knife edges of wrought iron at A, C, and B, whose distances AC and CB were each five feet. The angles of the knife edges

Now $\frac{1}{12}bc^3$ representing (Art. 364.) the moment of inertia of the rectangular section of the beam about an axis passing through its centre of gravity, it follows (Art. 79.) that the moment I about an axis parallel to this passing through a point at distance h from it is represented by

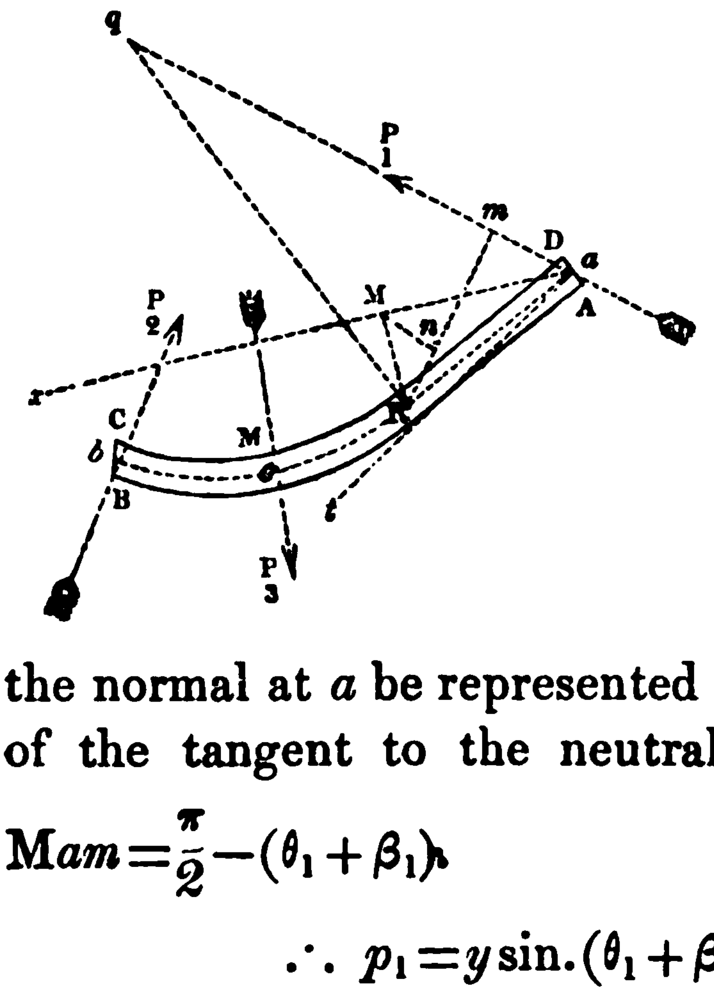
$$I = h^2bc + \frac{1}{12}bc^3.$$

Substituting, therefore, the value of h from equation (504),

$$I = \frac{R^2P_1^{\,2}}{E^2bc}\sin.^2\theta + \frac{1}{12}bc^3 \quad \ldots\ldots \quad (585).$$

Substituting this value in equation (505), and reducing,

$$\frac{1}{R} = \frac{12P_1Ebcp_1}{12R^2P^2\sin.^2\theta + E^2b^2c^4} \quad \ldots\ldots \quad (586).$$



Draw ax parallel to the position of the beam before deflexion; take this line as the axis of the abscissæ and a as the origin; then $p_1 = Rm = Rn + nm = \overline{MR}\cos. MRm + \overline{aM}\sin. Mam = y\cos. Mam + x\sin. Mam$.

Let, now, the inclination DaP_1 of the direction of P_1 to the normal at a be represented by θ_1, and the inclination Mat of the tangent to the neutral line at a to ax, by β_1; then

$Mam = \frac{\pi}{2} - (\theta_1 + \beta_1)$,

$$\therefore\ p_1 = y\sin.(\theta_1+\beta_1) + x\cos.(\theta_1+\beta_1).$$

Substituting this value of P_1 in the preceding equation,

$$\frac{1}{R} = \frac{12P_1Ebc\{y\sin.(\theta_1+\beta_1) + x\cos.(\theta_1+\beta_1)\}}{12R^2P_1^{\,2}\sin.^2\theta + Eb^2c^4} \quad \ldots\ldots \quad (587);$$

where θ represents (Art. 357.) the inclination Rqa of the normal at the point R to the direction of P_1.

384. *Case in which the deflexion of the beam is small.*

If the deflexion be small, and the inclination θ_1, of the direction of P_1 to the normal at its point of application, be not greater than $\frac{\pi}{4}$; then $y \sin.(\theta_1+\beta_1)$ is exceedingly small, and may be neglected as compared with $x \cos.(\theta_1+\beta_1)$; in this case, moreover, θ is, for all positions of R, very nearly equal to θ_1. Neglecting, therefore, β_1 as exceedingly small, we have

$$\frac{1}{R}=\frac{12P_1Ebcx\cos.\theta_1}{12R^2P_1^2\sin.^2\theta_1+E^2b^2c^4} \quad \dots \quad (588).$$

Solving this equation, of two dimensions, in respect to $\frac{1}{R}$, and taking the greater root,

$$\frac{1}{R}=\frac{6P_1}{Ebc^3}\{x\cos.\theta_1+\sqrt{x^2\cos.^2\theta_1-\tfrac{1}{3}c^2\sin.^2\theta_1}\} \quad \dots \quad (589).$$

385. THE WORK EXPENDED UPON THE DEFLEXION OF A UNIFORM RECTANGULAR BEAM, WHEN THE DEFLECTING PRESSURES ARE INCLINED AT ANY ANGLE GREATER THAN HALF A RIGHT ANGLE TO THE SURFACE OF THE BEAM.

If u_1 represent work expended on the deflexion of the portion AM of the beam, then (equation 510)

$$u_1=\frac{P_1^2}{2E}\int_0^{a_1}\frac{p_1^2}{I}dx;$$

but by equation (505) $\frac{p_1^2}{I}=\frac{E}{P_1}\cdot\frac{p_1}{R}$,

$$\therefore\ u_1=\tfrac{1}{2}P_1\int_0^{a_1}\frac{p_1}{R}dx \quad \dots \quad (590).$$

But $\frac{p_1}{R}=\frac{6P_1}{Ebc^3}\{x\cos.\theta_1+\sqrt{x^2\cos.^2\theta_1-\tfrac{1}{3}c^2\sin.^2\theta_1}\}\ x\ \cos.\theta_1$;

by equation (589), observing that the deflexion being small,

$p_1 = x\cos.\theta_1$ very nearly. Now the value of $\frac{1}{R}$ (equation 589) becomes impossible at the point where $x\cos.\theta_1$ becomes less than $\frac{1}{\sqrt{3}}c\sin.\theta_1$; the curvature of the neutral line commences therefore at that point, according to the hypotheses on which that equation is founded. Assuming, then, the corresponding value $\frac{1}{\sqrt{3}}c\tan.\theta_1'$ of x to be represented by x_1, the integral (equation 590) must be taken between the limits x_1 and a_1, instead of 0 and a_1;

$$\therefore u_1 = \frac{3P_1^2\cos.\theta_1}{Ebc^3}\int_{x_1}^{a_1}\{x^2\cos.\theta_1 + x\sqrt{x^2\cos.^2\theta_1 - \tfrac{1}{3}c^2\sin.^2\theta_1}\}dx;$$

$$\therefore u_1 = \frac{P_1^2\cos.^2\theta_1}{Ebc^3}\left\{a_1^3 - \frac{1}{3\sqrt{3}}c^3\tan.^3\theta_1 + (a_1^2 - \tfrac{1}{3}c^2\tan.^2\theta_1)^{\frac{3}{2}}\right\} \quad ..(59$$

And a similar expression being evidently obtained for the work expended in the deflexion of the portion BM of the beam, it follows, neglecting the term involving c^3 as exceedingly small when compared with a_1^3, that the whole work U_1 expended upon the deflexion is represented by the equation

$$U_2 = \frac{1}{Ebc^3}\left\{P_1^2\cos.^2\theta_1\{a_1^3 + (a_1^2 - \tfrac{1}{3}c^2\tan.^2\theta_1)^{\frac{3}{2}}\} + P_2^2\cos.^2\theta_2\{a_2^3 + (a_2^2 - \tfrac{1}{3}c^2\tan.^2\theta_2)^{\frac{3}{2}}\}\right\}$$

But if θ_3 be taken to represent the inclination of P_3 to the normal to the surface of the beam, as θ_1 and θ_2 represent the similar inclinations of P_1 and P_2, then, the deflexion being small,

$$P_1 a\cos.\theta_1 = P_3 a_2\cos.\theta_3, \quad P_2 a\cos.\theta_2 = P_3 a_1\cos.\theta_3.$$

Eliminating P_1 and P_2 between these equations and the preceding,

$$U_2 = \frac{P_3^2\cos.^2\theta_3}{Ea^2bc^3}\left\{a_2^2\{a_1^3 + (a_1^2 - \tfrac{1}{3}c^2\tan.^2\theta_1)^{\frac{3}{2}}\} + a_1^2\{a_2^3 + (a_2^2 - \tfrac{1}{3}c^2\tan.^2\theta_2)^{\frac{3}{2}}\}\right\} \quad ..(59$$

If the pressure P_3' be applied perpendicularly in the centre of the beam, and the pressures P_1 and P_2 be applied at its extremities in directions equally inclined to its surface; then

$a_1=a_2=\frac{1}{2}a$, $\theta_1=\theta_2=\theta$, and $\theta_3=0$. Substituting these values in the preceding equations, and reducing,

$$U_3=\frac{P_3^2\{a^3+(a^2-\frac{4}{3}c^2\tan.^2\theta)^{\frac{3}{2}}\}}{16Ebc^3} \quad \ldots . (593).$$

386. The linear deflexion of a rectangular beam.

D_1 being taken as before (Art. 370.) to represent the deflexion of the extremity A measured in a direction perpendicular to the surface of the beam, we have (Art. 52.)

$$u_1=\int P_1\cos.\theta_1 dD_1,$$

$$\therefore\ P_1\cos.\theta=\frac{du_1}{dD_1}=\frac{du_1}{dP_1}\cdot\frac{dP_1}{dD_1}.$$

But by equation (591),

$$\frac{du_1}{dP_1}=\frac{2P_1}{Ebc^3}\cos.^2\theta_1\{a_1^3+(a_1^2-\tfrac{1}{3}c^2\tan.^2\theta_1)^{\frac{3}{2}}\}.$$

$$\therefore\ P_1\cos.\theta_1=\frac{dP_1}{dD_1}\cdot\frac{2P_1}{Ebc^3}\cos.^2\theta_1\{a_1^3+(a_1^2-\tfrac{1}{3}c^2\tan.^2\theta_1)^{\frac{3}{2}}\}.$$

Dividing both sides by P_1, reducing, and integrating,

$$D_1=\frac{2P_1}{Ebc^3}\cos.\theta_1\{a_1^3+(a_1^2-\tfrac{1}{3}c^2\tan.^2\theta_1)^{\frac{3}{2}} \quad \ldots (594).$$

Proceeding similarly in respect to the deflexion D_3 perpendicular to the surface of the beam at the point of application of P_3, we obtain from equation (592)

$$D_3=\frac{2P_3\cos.\theta_3}{Ea^2bc^3}\left\{a_2^2\{a_1^3+(a_1^2-\tfrac{1}{3}c^2\tan.^2\theta_1)^{\frac{3}{2}}\}+a_1^2\{a_2^3+(a_2^2-\tfrac{1}{3}c^2\tan.^2\theta_2)^{\frac{3}{2}}\}\right\} \quad \ldots (595).$$

In the case in which P_1 and P_2 are equally inclined to the extremities of the beam and the direction of P_3 bisects it, this equation becomes

$$D_3=\frac{P_3\{a^3+(a^2-\frac{4}{3}c^2\tan.^2\theta)^{\frac{3}{2}}\}}{8Ebc^3} \quad \ldots . (596).$$

weight μs of DP, it has been shown (Art. 395.) that DQ will represent the tension c at D, and TQ that at P.

Also, $\frac{dy}{dx} = \tan. PQM = \tan. DQT = \frac{DT}{DQ} = \frac{\mu s}{c}$,

$$\therefore \frac{dy}{dx} = \frac{\mu s}{c} \quad \ldots \quad (605).$$

Again, $\frac{dx}{ds} = \left(1 + \frac{dy^2}{dx^2}\right)^{-\frac{1}{2}} = \left(1 + \frac{\mu^2 s^2}{c^2}\right)^{-\frac{1}{2}}$. Integrating between the limits 0 and s*, and observing that when $s=0$, $x=0$,

$$x = \frac{c}{\mu} \log._{\varepsilon} \left\{ \frac{\mu s}{c} + \left(1 + \frac{\mu^2 s^2}{c^2}\right)^{\frac{1}{2}} \right\} \quad \ldots \ldots \quad (606).$$

$$\therefore \frac{\mu s}{c} + \left(1 + \frac{\mu^2 s^2}{c^2}\right)^{\frac{1}{2}} = \varepsilon^{\frac{\mu x}{c}},$$

$$\therefore \frac{\mu s}{c} - \left(1 + \frac{\mu^2 s^2}{c^2}\right)^{\frac{1}{2}} = -\varepsilon^{\frac{-\mu x}{c}}.$$

By addition and reduction,

$$s = \frac{1}{2}\frac{c}{\mu}\left(\varepsilon^{\frac{\mu x}{c}} - \varepsilon^{\frac{-\mu x}{c}}\right) \quad \ldots \quad (607).$$

Substituting this value for s in equation (605), and integrating between the limits 0 and x,

$$y = \frac{1}{2}\frac{c}{\mu}\left(\varepsilon^{\frac{\mu x}{c}} + \varepsilon^{\frac{-\mu x}{c}}\right) \quad \ldots \quad (608);$$

which is the equation to the catenary.

397. *The tension* (c) *on the lowest point of the catenary.*

Let 2S represent the whole length of the chain, and $2a$ the horizontal distance between the points of attachment. Now when $x=a$, $s=S$; therefore (equation 607),

* See Hymer's Int. Cal., art. 15.

$$S=\frac{1}{2}\frac{c}{\mu}\left(\varepsilon^{\frac{\mu a}{c}}-\varepsilon^{\frac{-\mu a}{c}}\right)\;.\;.\;.\;.\;(609);$$

from which expression the value of c may be determined by approximation.

398. *The tension at any point of the chain.*

The tension T at P is represented by $\overline{TQ}=\sqrt{\overline{DQ}^2+\overline{DT}^2}$;

$$\therefore\; T=(c^2+\mu^2 s^2)^{\frac{1}{2}}\;.\;.\;.\;.\;.\;(610).$$

Now the value of c has been determined in the preceding article; the tension upon any point of the chain whose distance from its lowest point is s is therefore known.

399. *The inclination of the curve to the vertical at any point.*

Let ι represent this inclination, then $\cot.\iota=\frac{dy}{dx}$;

$$\therefore\;(\text{equation } 608)\;\cot.\iota=\frac{1}{2}\left(\varepsilon^{\frac{\mu x}{c}}-\varepsilon^{\frac{-x\mu}{c}}\right)\;.\;.\;.\;.\;(611).$$

The inclination may be determined without having first determined the value of c, by substituting $\cot.\iota$ for $\frac{\mu s}{c}$ in equation (606); we thus obtain, writing also a and S for x and s,

$$\frac{a}{S}=\tan.\iota\log._{\varepsilon}(\cot.\iota+\text{cosec}.\iota)=\tan.\iota\log._{\varepsilon}\cot.\tfrac{1}{2}\iota;$$

$$\therefore\;-\tan.\iota\log._{\varepsilon}\tan.\tfrac{1}{2}\iota=\frac{a}{S}\;.\;.\;.\;.\;.\;(612).$$

This equation may readily be solved by approximation; and the value of c may then be determined by the equation $c=\mu S\tan.\iota$.

Integrating these expressions *, we obtain

$$x=\frac{\tau c}{m\mu_1}\left(c^2+\frac{\tau c\mu_2}{m\mu_1}\right)^{-\frac{1}{2}}\tan.^{-1}\left(c^2+\frac{\tau c\mu_2}{m\mu_1}\right)^{-\frac{1}{2}}u \quad \ldots\ldots \quad (622).$$

$$y=\frac{\tau}{2m\mu_1}\log._{\epsilon}\left\{\frac{u^2+c^2+\frac{\tau c\mu_2}{m\mu_1}}{c^2+\frac{\tau c\mu_2}{m\mu}}\right\}.$$

Substituting in this equation the value of u given by the preceding equation, and reducing,

$$y=\frac{\tau}{m\mu_1}\log._{\epsilon}\sec.\left\{\frac{m\mu_1}{\tau}\left(1+\frac{\tau\mu_2}{cm\mu_1}\right)^{\frac{1}{2}}x\right\} \quad \ldots\ldots \quad (623);$$

which is the equation to the suspension chain of uniform strength, and therefore OF THE GREATEST STRENGTH WITH A GIVEN QUANTITY OF MATERIAL.

403. *To determine the variation of the section* K *of the chain of the suspension bridge of the greatest strength.*

Let the value of u determined by equation (622) be substituted in equation (621); we shall thus obtain by reduction

$$K=\frac{mc}{\tau}\left\{1+\left(1+\frac{\tau\mu_2}{mc\mu_1}\right)\tan.^2\frac{m\mu_1}{\tau}\left(1+\frac{\tau\mu_2}{cm\mu_1}\right)x\right\}^{\frac{1}{2}} \ldots (624).\dagger$$

It is evident from this expression that the area of the section of the chains, of the suspension bridge of uniform strength, and therefore of the greatest economy of material, increases from the lowest point towards the points of suspension, where it is greatest.‡

* Hall's Diff. Cal. pp. 280, 283.

† $\frac{ds}{dx}=\frac{\tau K}{mc}$; ∴ $s=\frac{\tau}{\mu c}\int K dx$. Now the function K (equation 624) may be integrated in respect to x by known rules of the integral calculus; the value of s may therefore be determined in terms of x, and thence the length in terms of the span. The formula is omitted by reason of its length.

‡ This variation of the section of the chains is exhibited in a suspension bridge recently invented by Mr. Dredge, and appears to constitute the whole merit of that invention.

contained in it may be precisely equal in weight to the material of the suspending rods. It is evident that the conditions of the equilibrium will, on this hypothesis, be very nearly the same as in the actual case. Let μ_3 represent the weight of each square foot of this plate, then will $\mu_3\int y\,dx$ represent the weight of that portion of it which is suspended from the portion DP of the chain, and the whole load u upon that portion of the chain will be represented by

$$u=\mu_1\int \mathrm{K}\,ds+\mu_2 x+\mu_3\int y\,dx \quad \ldots\ldots \quad (627).$$

It may be shown, as before (Art. 402.), that

$$\frac{dy}{dx}=\frac{u}{c},\quad \mathrm{K}\tau=m(c^2+u^2)^{\frac{1}{2}} \quad \ldots\ldots \quad (628),$$

$\int \mathrm{K}\,ds=\frac{m}{\tau c}\int(c^2+u^2)dx.$ Substituting in equation (627), differentiating in respect to x, and observing that $\frac{du}{dx}=\frac{u}{c}\frac{du}{dy}$,

$$\frac{du}{dx}=\frac{u}{c}\frac{du}{dy}=\frac{m\mu_1}{\tau c}(c^2+u^2)+\mu_2+\mu_3 y \quad \ldots\ldots \quad (629).$$

Transposing, reducing, and assuming,

$$\frac{m\mu_1}{\tau}=\alpha \quad \ldots\ldots \quad (630);$$

$$\frac{du^2}{dy}-2\alpha u^2=2c(\mu_3 y+\alpha c+\mu_2).$$

A linear equation in u^2, the integration of which by a well-known method (Hall's *Diff. Cal.* p. 397.) gives

$$u^2\epsilon^{-2\alpha y}=2c\int(\mu_3 y+\alpha c+\mu_2)\epsilon^{-2\alpha y}dy+\mathrm{C}.$$

Assuming the length of the shortest connecting rod DC to be represented by b, integrating between the limits b and y, and observing that when $y=b$, $u=0$,

$$u^2\epsilon^{-2\alpha y}=\frac{c}{\alpha}\left\{\mu_3\left(b\epsilon^{-2\alpha b}-y\epsilon^{-2\alpha y}\right)+\left(\frac{\mu_3}{2\alpha}+\alpha c+\mu_2\right)\left(\epsilon^{-2\alpha}-\epsilon^{-2\alpha}\right)\right\};$$

[redacted]

Extracting the square root of both sides, transposing, and integrating,

$$x^2=\left(\frac{2c}{\mu_3 b+\alpha c+\mu_2}\right)(y-b) \quad . \; . \; . \; . \; . \quad (633);$$

the equation to a parabola whose vertex is in D, and its axis vertical.

The values a and H of x and y at the points of suspension being substituted in this equation, and it being solved in respect to c, we obtain

$$c=\left(\frac{\mu_2+\mu_3 b}{2\mathrm{H}-2b-\alpha a^2}\right)a^2 \quad . \; . \; . \; . \; . \quad (634);$$

by which expression the tension c upon the lowest point of the curve is determined, and thence the length y of the suspending rod at any given distance x from the centre of the span, by equation (633), and the section K of the chain at that point by equation (632), which last equation gives by a reduction similar to the above

$$\mathrm{K}=\frac{\alpha c}{\mu_1}\left\{2(y-b)\left(\frac{\mu_3 b+\mu_2}{c}+\alpha\right)+1\right\}^{\frac{1}{2}} \quad . \; . \; . \; . \; . \quad (635).$$

407. ***The section of the chains being of uniform dimensions, as in the common suspension bridge, it is required to determine the conditions of the equilibrium.****

The weight of the suspending rods being neglected, and the same notation being adopted as in the preceding articles, except that μ_1 is taken to represent the weight of one foot in the length of the chains instead of a bar one square inch in section, we have by equation (619), since K is here constant,

$$u=\mu_1 s+\mu_2 x \quad . \; . \; . \; . \; . \quad (636).$$

Differentiating this equation in respect to x, and observing

* This problem appears first to have been investigated by Mr. Hodgkinson in the fifth volume of the Manchester Transactions; his investigation extends to the case in which the influence of the weights of the suspending rods is included.

that $\frac{ds}{dx}=\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}=\left(1+\frac{u^2}{c^2}\right)^{\frac{1}{2}}$ (equation 620), and that

$$\frac{du}{dx}=\frac{du}{dy}\frac{dy}{dx}=\frac{du}{dy}\frac{u}{c}=\mu_1\left(1+\frac{u^2}{c^2}\right)^{\frac{1}{2}}+\mu_2;$$

$$\therefore\ x=\int_0^u\frac{cdu}{\mu_1(c^2+u^2)^{\frac{1}{2}}+\mu_2c},\ y=\int_0^u\frac{udu}{\mu_1(c^2+u^2)^{\frac{1}{2}}+\mu_2c}.$$

The former of these equations may be rationalised by assuming $(c^2+u^2)^{\frac{1}{2}}=c+zu$, and the latter by assuming $(c^2+u^2)^{\frac{1}{2}}=z$; there will thus be obtained by reduction

$$x=2c\int_0^z\frac{(1+z^2)dz}{(1-z^2)\{(\mu_1+\mu_2)+(\mu_1-\mu_2)z^2}, y=\int_c^z\frac{zdz}{\mu_1z+\mu_2c}.$$

The latter equation may be placed under the form

$$y=\frac{1}{\mu_1}\int_c^z\left\{1-\frac{c\mu_2}{\mu_1z+\mu_2c}\right\}dz;$$

which expression being integrated and its value substituted for z, we obtain

$$y=\frac{1}{\mu_1}\left\{(c^2+u^2)^{\frac{1}{2}}-c-\frac{c\mu_2}{\mu_1}\log._{\epsilon}\frac{\mu_1(c^2+u^2)^{\frac{1}{2}}+\mu_2c}{(\mu_1+\mu_2)c}\right\}\ \ldots (637).$$

The method of rational fractions (Hymer's *Integ. Calc.* § 2.) being applied to the function under the integral sign in the former equation, it becomes

$$x=\frac{2c}{\mu_1}\int_0^z\left\{\frac{1}{1-z^2}-\frac{\mu_2}{(\mu_1-\mu_2)z^2+(\mu_1+\mu_2)}\right\}dz.$$

The integral in the first term in this expression is represented by $\frac{1}{2}\log._{\epsilon}\left(\frac{1+z}{1-z}\right)$, and that of the second term by

$$\frac{\mu_2}{(\mu_1^2-\mu_2^2)^{\frac{1}{2}}}\tan.^{-1}\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)^{\frac{1}{2}}z, \text{ or } \frac{\mu_2}{2(\mu_2^2-\mu_1^2)^{\frac{1}{2}}}\log._{\epsilon}\frac{(\mu_2+\mu_1)^{\frac{1}{2}}+(\mu_2-\mu_1)^{\frac{1}{2}}z}{(\mu_2+\mu_1)^{\frac{1}{2}}-(\mu_2-\mu_1)^{\frac{1}{2}}z},$$

according as μ_1 is greater or less than μ_2, or according as the

mass upon another were accurately the same in every direction in which the plane CD may be imagined to intersect the mass, then would the plane of actual rupture be inclined to the base at an angle represented by the formula

$$\alpha=\frac{\pi}{4}+\frac{\varphi}{2} \quad . \quad . \quad . \quad . \quad . \quad (640);$$

since the value of P would in this case be (equation 639) a minimum when sin. $(2\alpha-\varphi)$ is a maximum, or when $2\alpha-\varphi=\frac{\pi}{2}$, or $\alpha=\frac{\pi}{4}+\frac{\varphi}{2}$; whence it follows that a plane inclined to the base at that angle is that plane along which the rupture will first take place, as P is gradually increased beyond the limits of resistance.

The actual inclination of the plane of rupture was found in the experiments of Mr. Hodgkinson to vary with the material of the column. In cast iron, for instance, it varied according to the quality of the iron from 48° to 58° *, and was different in different species. By this dependence of the angle of rupture upon the nature of the material, it is proved that the value of the modulus of sliding coherence γ is not the same for every direction of the plane of rupture, or that the value of φ varies greatly in different qualities of cast iron.

Solving equation (639) in respect to γ, we obtain

$$\gamma=\frac{P}{K}\sin.(\alpha-\varphi)\cos.\alpha\sec.\varphi \quad . \quad . \quad . \quad . \quad . \quad (641);$$

from which expression the value of the modulus γ may be determined in respect to any material whose limiting angle of resistance φ is known, the force P producing rupture, under the circumstances supposed, being observed, and also the angle of rupture.†

* Seventh Report of British Association, p. 349.

† A detailed statement of the results obtained in the experiments of Mr. Hodgkinson on this subject is contained in the Appendix to the "Illustrations of Mechanics" by the author of this work.

The Section of Rupture in a Beam.

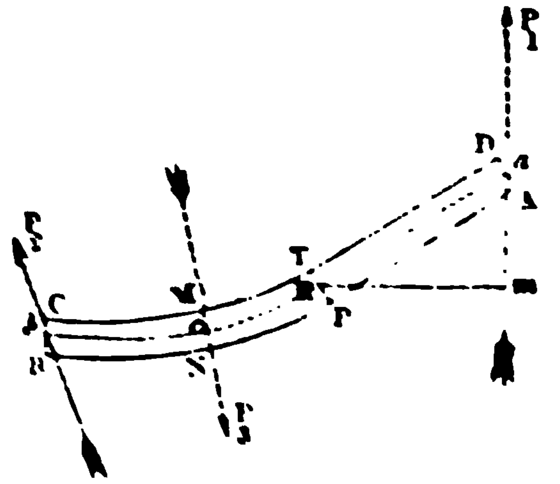
409. When a beam is deflected under a transverse strain, the material on that side of it on which it sustains the strain is compressed, and the material on the opposite side extended. That imaginary surface which separates the compressed from the extended portion of the material is called its neutral surface (Art. 356.), and its position has been determined under all the ordinary circumstances of flexure. That which constitutes the strength of a beam is the resistance of its material to compression on the one side of its neutral surface, and to extension on the other; so that if either of these yield the beam will be broken.

The *section of rupture* is that transverse section of the beam about which, in its state bordering upon rupture, it is the *most* extended, if it be about to yield by the extension of its material, or the *most* compressed if about to yield by the compression of its material.

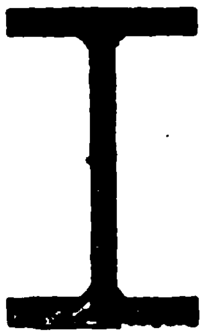
In a prismatic beam, or a beam of uniform dimensions, it is evidently that section which passes through the point of greatest curvature of the neutral line, or the point in respect to which the radius of curvature of the neutral line is the least, or its reciprocal the greatest.

General Conditions of the Rupture of a Beam.

410. Let PQ be the section of rupture in a beam sustaining any given pressures, whose resultants are represented, if they be more in number than three, by the three pressures P_1, P_2, P_3. Let the beam be upon the point of breaking by the yielding of its material to extension at the point of greatest extension P; and let R represent, in the state of the beam bordering upon rupture, the intersection of the neutral sur-



material is shown collected in two thin but wide flanges, but united by a narrow rib.



That which constitutes the strength of the beam being the resistance of its material to compression on the one side of its neutral axis, and its resistance to extension on the other side, it is evidently (Art. 390.) a second condition of the strongest form of any given section that when the beam is about to break across that section by extension on the one side, it may be about to break by compression on the other. So long, therefore, as the distribution of the material is not such as that the compressed and extended sides would yield together, the strongest form of section is not attained. Hence it is apparent that the strongest form of the section collects the greater quantity of the material on the compressed or the extended side of the beam, according as the resistance of the material to compression or to extension is the less. Where the material of the beam is cast iron *, whose resistance to extension is greatly less than its resistance to compression, it is evident that the greater portion of the material must be collected on the extended side.

Thus then it follows, from the preceding condition and this, that the strongest form of section in a cast iron beam is that by which the material is collected into two unequal flanges joined by a rib, the greater flange being on the extended side; and the proportion of this inequality of the flanges being just such as to make up for the inequality of the resistances of the material to rupture by extension and compression respectively.

Mr. Hodgkinson, to whom this suggestion is due, has directed a series of experiments to the determination of that proportion of the flanges by which the strongest form of section is obtained.†

* It is only in cast iron beams that it is customary to seek an economy of the material in the strength of the section of the beam ; the same principle of economy is surely, however, applicable to beams of wood.

† Memoirs of Manchester Philosophical Society, vol. iv. p. 453. Illustrations of Mechanics, Art. 68.

the particular section across which rupture will actually take place is that in respect to which equation (642) is *first* satisfied, as P_1 is continually increased; or that section in respect to which the formula

$$\frac{I}{p_1 c_1} \quad \ldots \ldots \quad (643)$$

is the least.

If the beam be loaded along its whole length, and x represent the distance of any section from the extremity at which the load commences, and μ the load on each foot of the length, then (Art. 373.) $P_1 p_1$ is represented by $\frac{1}{2}\mu x^2$. The section of rupture in this case is therefore that section in respect to which μ is first made to satisfy the equation $\frac{1}{2}\mu x^2 = \frac{SI}{c_1}$; or in respect to which the formula

$$\frac{I}{x^2 c_1} \quad \ldots \ldots \quad (644)$$

is the least.

If the section of the beam be uniform, $\frac{I}{c_1}$ is constant; the section of rupture is therefore evidently that which is most distant from the free extremity of the beam.

415. The beam of greatest strength.

The beam of greatest strength being that (Art. 390.) which presents an equal liability to rupture across every section, or in respect to which every section is brought into the state bordering upon rupture by the same deflecting pressure, is evidently that by which a given value of P is made to satisfy equation (642) for all the possible values of I, p_1, and c_1, or in respect to which the formula

$$\frac{I}{p_1 c_1} \quad \ldots \ldots \quad (645)$$

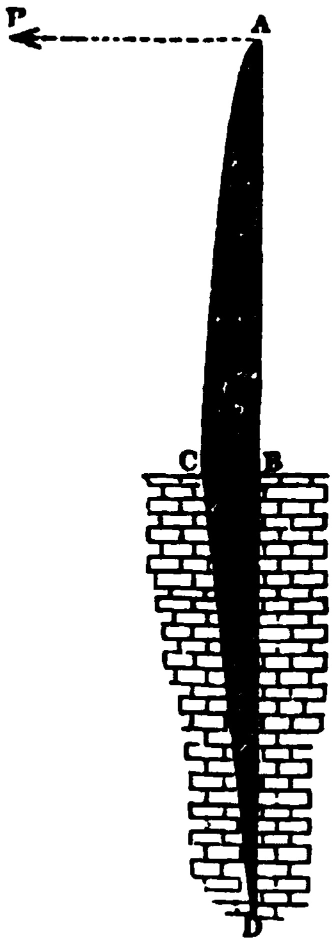
is constant.

[redacted]

$I=\frac{1}{4}\pi(r_1^4-r_2^4)$; which expression may be put under the form $\pi cr(r^2+\frac{1}{4}c^2)$ (see Art. 86.), r representing the mean radius of the hollow cylinder, and c its thickness. Also $c_1=r_1=r+\frac{1}{2}c$;

$$\therefore\ P=\pi S\frac{(r^2+\frac{1}{4}c^2)cr}{(r+\frac{1}{2}c)a} \quad \ldots\ldots \quad (650).$$

417. *The strongest form of beam under the conditions supposed in the last article.*

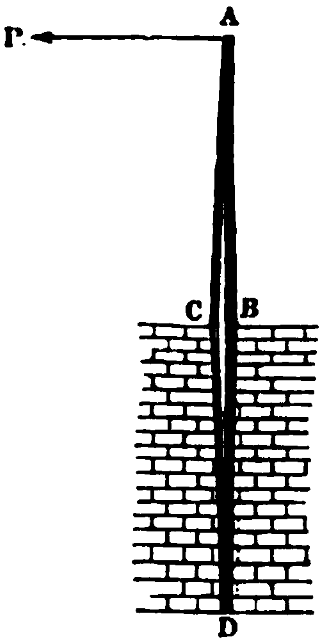


1st. Let the section of the beam be a rectangle, and let y be the depth of this rectangle at a point whose distance from its extremity A is represented by x, and let its breadth b be the same throughout. In this case $I=\frac{1}{12}by^3$, $c_1=\frac{1}{2}y$; therefore (equation 642) $P=\frac{SI}{c_1x}=\frac{1}{6}Sb\frac{y^2}{x}$. If, therefore, P be taken to represent the pressure which the beam is destined just to support, then the form of its section ABC is determined (Art. 415.) by the equation

$$y^2=\frac{6P}{Sb}x \quad \ldots\ldots \quad (651);$$

it is therefore a parabola, whose vertex is at A.*

If the portion DC of the beam do not rest against masonry at every point, but only at its extremity D, its form should evidently be the same with that of ABC.



2d. Let the section be *a circle*, and let y represent its radius at distance x from its extremity A, then $I=\frac{1}{4}\pi y^4$, $c_1=y$; therefore $P=\frac{1}{4}\pi S\frac{y^3}{x}$ so that the geometrical form of its longitudinal in section is determined by the equation

* The portion of the beam imbedded in the masonry should have the form described in Art. 419.

$$y^3=\frac{4P}{\pi S}x \quad \ldots\ldots \quad (652),$$

P representing the greatest pressure to which it is destined to be subjected.

418. The conditions of the rupture of a beam supported at one extremity, and loaded throughout its whole length.

Representing the weight resting upon each inch of its length a by w, and observing that the moment of the weight upon a length x of the beam from A, about the corresponding neutral axis, is represented (Art. 373.) by $\frac{1}{2}\mu x^2$, it is apparent (Art. 414.) that, if the beam be of uniform dimensions, its section of rupture is BD. Its strength is determined by substituting $\frac{1}{2}\mu a^2$ for P_1p_1 in equation (642), and solving in respect to μ; we thus obtain

$$\mu=\frac{2SI}{c_1a^2} \quad \ldots\ldots \quad (653);$$

by which equation is determined the uniform load to which the beam may be subjected, on each inch of its length.

For a rectangular beam, whose width is b and its depth c, this expression becomes

$$\mu=\frac{Sbc^2}{3a^2} \quad \ldots\ldots \quad (654).$$

419. To determine the form of greatest strength (Art. 415.)

in the case of a beam having a rectangular section of uniform breadth, $\frac{1}{2}\mu x^2$ must be substituted for $P_1 p_1$ in equation (642), and $\frac{1}{12}by^3$ for I, and $\frac{1}{2}y$ for c_1; whence we obtain by reduction

$$y=\left(\frac{3\mu}{Sb}\right)^{\frac{1}{2}}x \quad . \; . \; . \; . \; . \quad (655).$$

The form of greatest strength is therefore, in this case, the straight line joining the points A and B; the distance DB being determined by substituting the distance AD for x in the above equation.

That portion BED of the beam which is imbedded in the masonry should evidently be of the same form with DBA.*

420. If, in addition to the uniform load upon the beam, a given weight W be suspended from A, $\frac{1}{2}\mu x^2+Wx$ must be substituted for $P_1 p_1$ in equation (642); we shall thus obtain for the equation to the form of greatest strength

$$y^2=\frac{3\mu}{Sb}\left\{\frac{2W}{\mu}x+x^2\right\} \quad . \; . \; . \; . \; . \quad (656),$$

which is the equation to an hyperbola having its vertex at A.

* It is obvious that in all cases the strength of a beam at each point of its length is dependent upon the dimensions of its cross section at that point, and that its general form may in any way be changed without impairing its strength, provided those dimensions of the section be everywhere preserved.

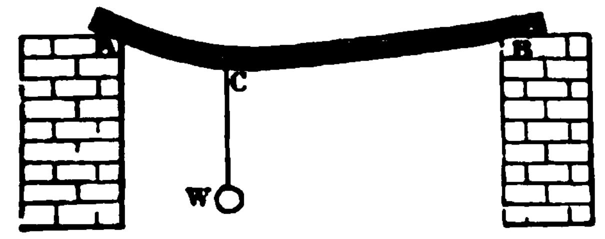
$$\therefore \frac{6\mu}{\mathrm{SA}_1}x^2 = \frac{(1+n)(d_1^2+nd_2^2)}{y} + 12ny \quad \ldots\ldots (658).$$

If the flanges be exceedingly thin, d_1 and d_2 are exceedingly small; the first term of this expression may therefore be neglected. The equation will then become that to a parabola whose vertex is at A and its axis vertical. This may therefore be assumed as a near approximation to the true form of the curve AQC.

Where the material is cast iron, it appears by Mr. Hodgkinson's experiments (Art. 413.) that n is to be taken $=6$.

422. A BEAM OF UNIFORM SECTION IS SUPPORTED AT ITS EXTREMITIES AND LOADED AT ANY POINT BETWEEN THEM: IT IS REQUIRED TO DETERMINE THE CONDITIONS OF RUPTURE.

The point of rupture in the case of a uniform section is



evidently (Art. 414.) the point C, from which the load is suspended; representing AB, AC, BC, by a, a_1, and a_2; and observing that the pressure P_1 upon the point B of the beam $=\frac{\mathrm{W}a_1}{a}$, so that the moment of P_1, in respect to the section of rupture $\mathrm{C}=\frac{\mathrm{W}a_1a_2}{a}$, we have, by equation (642), $\frac{\mathrm{W}a_1a_2}{a}=\frac{\mathrm{SI}}{c_1}$;

$$\therefore \mathrm{W}=\frac{\mathrm{SI}a}{a_1a_2c_1} \quad \ldots\ldots (659).$$

If the beam be *rectangular*, $\mathrm{I}=\frac{1}{12}bc^3$, $c_1=\frac{1}{2}c$,

$$\therefore \mathrm{W}=\frac{\mathrm{S}}{6}\,\frac{bc^2a}{a_1a_2} \quad \ldots\ldots (660);$$

where W represents the breaking weight, S the modulus of rupture, a the length, b the breadth, c the depth, and a_1, a_2 the distances of the point c from the two extremities, all these dimensions being in inches.

If the load be suspended in the middle, $a_1=a_2=\frac{1}{2}a$,

$$\therefore \mathrm{W}=\frac{2\mathrm{S}}{3}\,\frac{bc^2}{a} \quad \ldots\ldots (661).$$

$$\frac{Wa_2}{a}x=\frac{SI}{c_1} \quad \ldots\ldots \quad (665).$$

1st. Let the section be *rectangular;* let its breadth b be constant; and let its depth at the distance x from A be represented by y; therefore $I=\frac{1}{12}by^3$, $c_1=\frac{1}{2}y$. Substituting in the above equation and reducing,

$$y^2=\frac{6Wa_2}{Sab}x \quad \ldots\ldots \quad (666).$$

The curve AC is therefore a parabola, whose vertex is at A, and its axis horizontal. In like manner the curve BC is a parabola, whose equation is identical with the above, except that a_1 is to be substituted in it for a_2.

2d. Let the section of the beam be a circle. Representing the radius of a section at distance x from A by y, we have $I=\frac{1}{4}\pi y^4$, $c_1=y$; therefore by equation (665)

$$y^3=\frac{4Wa_2}{S\pi a}x \quad \ldots\ldots \quad (667).$$

3d. Let the section of the beam be circular; but let it be hollow, the thickness of its material being every where the same, and represented by c. If $y=$ mean radius of cylinder at distance x from A, then $I=\pi cy(y^2+\frac{1}{4}c^2)$, $c_1=(y+\frac{1}{2}c)$;

$$\therefore\ x=\frac{S\pi acy}{2Wa_2}\left(\frac{4y^2+c^2}{2y+c}\right) \quad \ldots\ldots \quad (668).$$

424. The beam of greatest absolute strength when loaded at a given point and supported at the extremities.

Let the section of the beam be that of greatest strength (Art. 413.). Substituting in equation (665) the value of $\frac{I}{c_1}$, as before in equation (657), and reducing,

$$\frac{6Wa_2x}{Sa}=\frac{(A_1d_1^2+A_2d_2^2+cy^3)(A_1+A_2+cy)+12A_1A_2y^2+3(A_1+A_2)cy^3}{cy^2+2A_1y}. \quad (669).$$

$$x(2a-x)=\frac{2n\mathrm{A}_1\mathrm{S}}{\mu}y \quad . \; . \; . \; . \; . \quad (676);$$

the equation to a parabola, whose axis is the vertical passing through the centre of the beam whose parameter is $\frac{2n\mathrm{A}_1\mathrm{S}}{\mu}$, and the position of its vertex D determined by the formula

$$\overline{\mathrm{CD}}=\frac{\mu a^2}{2n\mathrm{A}_1\mathrm{S}} \quad . \; . \; . \; . \; . \quad (677).$$

4. If it be proposed to make the rib or plate uniting the two flanges every where of the same depth *, and so to vary the breadths of the flanges as to give to the beam a uniform strength at all points under these circumstances; representing by y the breadth of the upper flange at a horizontal distance x from the point of support, we shall obtain, as in Art. 425.,

$$\frac{\mathrm{I}}{c_1}=y\frac{(n+1)}{12c}(d_1^2+nd_2^2)d_1+yncd_1.$$

Moreover, $\mathrm{P}_1p_1=\mu ax-\frac{1}{2}\mu x^2=\frac{1}{2}\mu x(2a-x)$; whence we obtain by substitution in equation (642), and reduction,

$$x(2a-x)=\left(\frac{\mathrm{S}d_1}{6\mu c}\right)\{(n+1)(d_1^2+nd_2^2)+12nc^2\}y \quad . \; . \; . \quad (678);$$

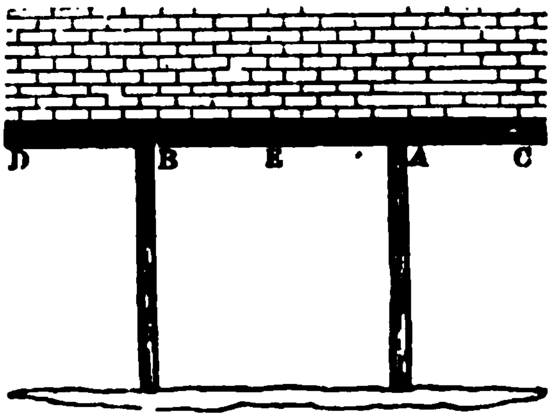
the equation to a parabola, whose axis is in the horizontal line bisecting the flange at right angles, its parameter represented by the coefficient of y in the preceding equation, and half the breadth of the flange in the middle determined by the formula

$$\frac{6ca^2\mu}{\{(n+1)(d_1^2+nd_2^2)+12nc^2\}\mathrm{S}d_1} \quad . \; . \; . \; . \; . \quad (679).$$

The equation to the lower flange is determined by substituting for y, in equation (678), $\frac{yd_2}{nd_1}$; whence it follows that the breadth of the lower flange in the middle is equal to that of the upper multiplied by the fraction $\frac{yd_2}{nd_1}$.

* As in Mr. Hodgkinson's construction.

427. A RECTANGULAR BEAM OF UNIFORM SECTION, AND UNIFORMLY LOADED THROUGHOUT ITS LENGTH, IS SUPPORTED BY TWO PROPS PLACED AT EQUAL DISTANCES FROM ITS EXTREMITIES: TO DETERMINE THE CONDITIONS OF RUPTURE.



It is evident from formula (644) that the section of rupture of the portion CA of the beam is at A, and therefore that the conditions of its rupture are determined (Art. 418.) by the equation

$$\mu_1=\frac{Sbc^2}{3a_1^2} \quad \ldots\ldots \quad (680);$$

where μ_1 represents, as before, the load upon each inch of the length of the beam, b its breadth, c its depth, and a_1 the length of the portion AC.

Again, it is evident that the point of rupture of the portion AB of the beam is at E. Now the value of P_1p_1 (equation 642) is, in respect to the portion AE of the beam, $\mu_2a(a-a_1)-\frac{1}{2}\mu_2a^2$; $2a$ representing the whole length of the beam, μ_2 the load upon each inch of the length of the beam which would produce rupture at E, and therefore μ_2a the resistance of each prop in the state bordering upon rupture; also $\frac{I}{c_1}=\frac{1}{6}bc^2$. Whence, by equation (642), $\mu_2a(a-a_1)-\frac{1}{2}\mu_2a^2=\mu_2a(\frac{1}{2}a-a_1)=\frac{1}{6}bc^2S$;

$$\therefore \mu_2=\frac{Sbc^2}{3a(a-2a_1)} \quad \ldots\ldots \quad (681).$$

428. THE BEST POSITIONS OF THE PROPS.

If the load μ be imagined to be continually increased, it is evident that rupture will eventually take place at A or at E according as the limit represented by equation (680), or that represented by equation (681), is first attained, or according as μ_1 or μ_2 is the *less*.

of n. It is, however, to be observed that the symbol a represents in that equation the distance AE (*fig.* Art. 429.); and that if we take it to represent the distance BE in that or the accompanying figure, we must substitute $\frac{a}{1-n}$ for a in equation (690), since $a=\text{BE}=\text{AE}-\text{AB}=(1-n)\text{AE}$; so that $\text{AE}=\frac{a}{1-n}$. This substitution being made, equation (690) becomes

$$\mu_2=\tfrac{1}{3}\frac{Sbc^2}{a^2}\,\frac{(2n-3)(1-n)^2}{n^3-4(1-n)^3};$$

and substituting the value ·6202 for n, we obtain by reduction

$$\mu_2=\frac{Sbc^2}{a^2}\;\ldots\ldots\ldots\;(692),$$

by which formula the load per inch of the length of the beam necessary to produce rupture is determined.

If the beam had not been prolonged beyond the points of support B and D and imbedded in the masonry, then the load per inch of the length necessary to produce rupture would have been represented by equation (674): eliminating between that equation and equation (692), we obtain $\mu_2=3\mu$; so that the load per inch of the length necessary to produce rupture is 3 times as great, when the extremities of the beam are prolonged and firmly imbedded in the masonry, as when they are free; i. e. *the strength of the beam is 3 times as great in the one case as in the other.*

432. THE STRENGTH OF COLUMNS.

For all the knowledge of this subject on which any

In all cases the strength of a column, one of whose ends was rounded and the other flat, was found to be an arithmetic mean between the strengths of two other columns of the same dimensions, one having both ends rounded and the other having both ends flat.

The above results only apply to the case in which the length of the column is so great that its fracture is produced wholly by the *bending* of its material; this limit is fixed by Mr. Hodgkinson in respect to columns of cast iron at about fifteen times the diameter when the extremities are rounded, and thirty times the diameter when they are flat. In shorter columns fracture takes place partly by the crushing and partly by the bending of the material. To these shorter columns the following rule was found to apply with sufficient accuracy: — "If W_1 represent the weight in tons which would break the column by bending alone (or if it did not crush) as given by the preceding formula, and W_2 the weight in tons which would break the column by crushing alone (or if it did not bend) as determined from the above table, then the actual breaking weight W of the column is represented in tons by the formula

$$W = \frac{W_1 W_2}{W_1 + \frac{3}{4} W_2} \cdot \cdot \cdot \cdot \cdot \cdot \quad (693).$$

Columns enlarged in the middle. — It was found that the strengths of columns of cast iron, whose diameters were from one and a half times to twice as great in the middle as at the extremities, were stronger by one seventh than solid columns, containing the same quantity of iron and of the same length, when their extremities were rounded; and stronger by one eighth or one ninth when their extremities were flat and rendered immoveable by discs.

433. Relative strength of long columns of cast iron, wrought iron, steel, and timber of the same dimensions. — Calling the strength of the cast iron column 1000, the strength of the wrought iron column will, according

[redacted]

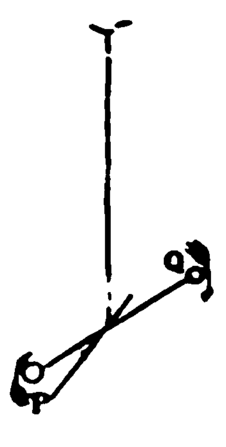
tion from the axis of the cylinder. Now the angle $a c \alpha$ or θ is evidently the *sum* of the angular displacements of all the sections between $a\alpha b\beta$ and AEB upon their subjacent sections; and the angular displacement of each upon its subjacent section is the same, the circumstances affecting the displacement of each being obviously the same: also the number of these sections varies as x, and the sum of their angular displacements is represented by θ; therefore the angular displacement of each section upon its subjacent section varies as $\frac{\theta}{x}$, and the actual displacement of the small element ΔK of the section $a\alpha b\beta$ varies as $\frac{\theta}{x}\rho$. Now the material being elastic, the pressure which must be applied to this element in order to keep it in this state of displacement varies as the amount of the displacement (Art. 347.), or as $\frac{\theta}{x}\rho$. Let its actual amount, when referred to a unit of surface, be represented by $G\frac{\theta}{x}\rho$, where G is a certain constant dependant for its amount on the elastic qualities of the material, and called the modulus of torsion; then will the force of torsion required to keep the element ΔK in its state of displacement be represented by $G\frac{\theta}{x}\rho\Delta K$, and its moment about the axis of the cylinder by $G\frac{\theta}{x}\rho^2\Delta K$. So that the sum of the moments of all such forces of torsion in respect to the whole section $a\alpha b\beta$ will be represented by $G\frac{\theta}{x}\Sigma\rho^2\Delta K$, or by $G\frac{\theta}{x}I$, if I represent the moment of inertia of the section about the axis of the cylinder. Now these forces are in equilibrium with P; therefore, by the principle of the equality of moments,

$$Pa = GI\frac{\theta}{x} \quad . \; . \; . \; . \quad (694).$$

If r represent the radius of the cylinder, $I=\frac{1}{2}\pi r^4$ (Art. 85.). Substituting this value, representing by L the whole

length of the cylinder, and by Θ the angle through which its extreme section CD is displaced or through which OP is made to revolve, called the *angle of torsion*, and solving in respect to Θ,

$$\Theta=\left(\frac{2a}{\pi G}\right)\cdot\frac{PL}{r^4}\quad\ldots\ldots\ (695).$$



Thus, then, it appears that when the dimensions of the cylinder are given, the angle of torsion Θ varies directly as the pressure P by which the torsion is produced; whence, also, it follows (Art. 97.) that if the cylinder, after having been deflected through any distance, be set free, it will oscillate isochronously about its position of repose, the time T of each oscillation being represented in seconds (equation 76) by the formula

$$T=\left(\frac{2\pi}{Gg}\right)^{\frac{1}{2}}\cdot\frac{a(WL)^{\frac{1}{2}}}{r^2}\quad\ldots\ldots\ (696),$$

since by equation (695) $P=\left(\frac{\pi r^4 G}{2a^2 L}\right)(\Theta a)$; in which expression (Θa) represents the length of the path described by the point P from its position of repose, so that the moving force upon the point P, when the pressure producing torsion is removed, varies as the path described by it from its position of repose.

The above is manifestly the theory of Coulomb's Torsion Balance.* W represents in the formula the weight of the mass supposed to be carried round by the point P, and the inertia of the cylinder itself is neglected as exceedingly small when compared with the inertia of this weight.

The torsion of rectangular prisms has been made the subject of the profound investigations of MM. Cauchy†, Lamé et Clapeyron‡, and Poisson.§ It results from these investi-

* Illustrations of Mechanics, art. 87.
† Exercises de Mathématique, 4e année.
‡ Crelle's Journal. § Memoires de l'Academie, tome viii.

[redacted]

gations* that if b and c be taken to represent the sides of the rectangular section of the prism, and the same notation be adopted in other respects as before, then

$$\Theta=\frac{3\mathrm{PL}a(b^2+c^2)}{\mathrm{G}b^3c^3} \; \ldots \; (697).$$

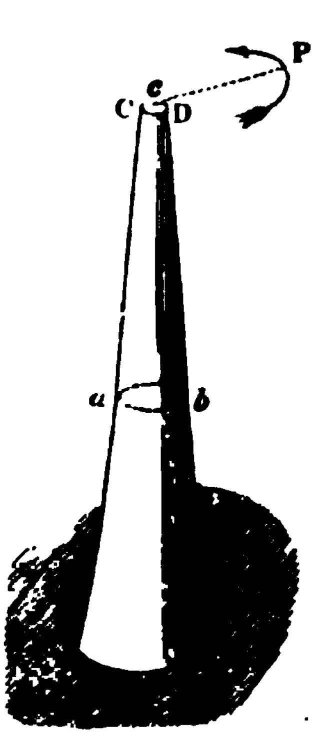
M. Cauchy has shown the values of the constant G to be related to those of the modulus of elasticity E by the formula

$$\mathrm{G}=\tfrac{2}{5}\mathrm{E} \; \ldots\ldots \; (698).$$

In using the values of G deduced by this formula from the table of moduli of elasticity, all the dimensions must be taken in inches, and the weights in pounds.

435. Elasticity of torsion in a solid having a circular section of variable dimensions.

Let ab represent an element of the solid contained by planes, perpendicular to the axis, whose distance from one another is represented by the exceedingly small increment Δx of the distance x of the section ab from the fixed section AB, and let its radius be represented by y; and suppose the whole of the solid except this single element to become rigid, a supposition by which the conditions of the equilibrium of this particular element will remain unchanged, the pressure P remaining the same, and being that which produces the torsion of this single element. Whence, representing by $\Delta\theta$ the angle of torsion of this element, and considering it a cylinder whose length is Δx, we have by equation (694), substituting for I its value $\frac{1}{2}\pi y^4$,



$$\mathrm{P}a=\tfrac{1}{2}\mathrm{G}\pi y^4\frac{\Delta\theta}{\Delta x}.$$

Passing to the limit, and integrating between the limits 0 and

* Navier, Resumé des Leçons, &c. Art. 159.

resistance upon any element ΔK by $\frac{T}{r}\rho\Delta K$, and the sum of the moments about the axis, of the resistances of all such elements, by $\frac{T}{r}\Sigma\rho^2\Delta K$, or by $\frac{T}{r}I$, or substituting for I its value (equation 64) by $\frac{1}{2}T\pi r^3$. But these resistances are in equilibrium with the pressure P, which produces torsion, acting at the distance a from the axis;

$$\therefore\ Pa=\tfrac{1}{2}T\pi r^3 \quad \ldots\ldots \quad (701).$$

It results from the researches of M. Cauchy, before referred to, that in the case of a rectangular section whose sides are represented by b and c, the conditions of rupture are determined by the equation

$$Pa^{*}=\tfrac{1}{3}T\frac{b^2c^2}{(b^2+c^2)^{\frac{1}{2}}} \quad \ldots\ldots \quad (702).$$

The *length* of a prism subjected to torsion does not affect the actual amount of the pressure required to produce rupture, but only the angle of torsion (equation 695) which precedes rupture, and therefore the space through which the pressure must be made to act, and the *amount of* WORK *which must be done* to produce rupture.

According to M. Cauchy, the modulus of rupture by torsion T is connected with that S of rupture by transverse strain by the equation

$$T=\tfrac{4}{5}S \quad \ldots\ldots \quad (703).$$

* Navier, Resumé d'un Cours, &c. Art. 167.

$\frac{W_1}{g}f_1$ represent (Art. 95.) the *effective* force upon W_1; and if f_2 be taken to represent under the same circumstances the increment of velocity received by W_2, then will $\frac{W_2}{g}f_2$ represent the effective force upon W_2. Whence it follows, by the principle of D'Alembert (Art. 103.), that if these effective forces be conceived to be applied to the bodies in directions opposite to those in which the corresponding retardation and acceleration take place, they will be in equilibrium with the other forces applied to the bodies. But, by supposition, no other forces than these are applied to the bodies: these are therefore in equilibrium with one another.

$$\therefore \frac{W_1}{g}f_1 = \frac{W_2}{g}f_2 \quad . \; . \; . \; . \; . \quad (704).$$

Let now an exceedingly small increment of the time from the commencement of the impact be represented by Δt, and let Δv_1 and Δv_2 represent the decrement and increment of the velocities of the bodies respectively during that time,

$$\therefore \text{(Art. 95.)} \; f_1 \Delta t = \Delta v_1, \; f_2 \Delta t = \Delta v_2;$$

$$\therefore \text{(equation 704)} \; W_1 \, . \, \Delta v_1 = W_2 \, . \, \Delta v_2;$$

and this equality obtaining throughout that period of the impact which precedes the period of greatest compression, it follows that when the bodies are moving in the same direction

$$W_1(V_1 - V) = W_2(V - V_2) \quad . \; . \; . \; . \; . \quad (705);$$

since $V_1 - V$ represents the whole velocity lost by W_1 during that period, and $V - V_2$ the whole velocity gained by W_2.

If the bodies be moving in opposite directions, and their common motion at the instant of greatest compression be in the direction of the motion of W_1, then is the velocity lost by W_1 represented as before by $(V_1 - V)$; but the sum of the decrements and increments of velocity communicated to W_2, in order that its velocity V_2 may in the first place be destroyed, and then the velocity V communicated to it in an opposite direction, is represented by $(V_2 + V)$,

the bodies in these two states of their motion has been expended in producing their compression; if, therefore, the amount of work thus expended be represented by u, we have

$$u=\tfrac{1}{2}\frac{W_1}{g}V_1^2+\tfrac{1}{2}\frac{W_2}{g}V_2^2-\tfrac{1}{2}\frac{W_1+W_2}{g}V^2;$$

or substituting for V its value from equation (706), and reducing,

$$u=\frac{1}{2g}\left(\frac{W_1W_2}{W_1+W_2}\right)(V_1\mp V_2)^2 \quad . \quad . \quad . \quad . \quad (708).$$

This expression represents the amount of work *permanently* lost in the impact of two *inelastic* bodies, their common velocity after impact being represented by equation (706). If W_2 be exceedingly great as compared with W_1,

$$u=\frac{W_1}{2g}(V_1\mp V_2)^2 \quad . \quad . \quad . \quad . \quad (709).$$

440. Two elastic bodies impinge upon one another: it is required to determine the velocity after impact.

If the impinging bodies be perfectly elastic, it is evident that after the period of their greatest compression is passed, they will, in the act of expanding their surfaces, exert mutual pressures upon one another, which are, in corresponding positions of the surfaces, precisely the same with those which they sustained whilst in the act of compression; whence it follows that the decrements of velocity experienced by that body whose motion is retarded by this expansion of the surfaces, and the increments acquired by that whose velocity is accelerated, will be equal to those before received in passing through corresponding positions, and therefore the whole decrements and increments thus received during the whole expansion equal to those received during the whole compression.

Now the velocity lost by W_1 during the compression is represented by (V_1-V); that lost by it during the expansion, or from the period of greatest compression to that when the

let them bear to one another the ratio of 1 to e. Now the velocity lost during compression by W_1 is under all circumstances represented by (V_1-V); that lost during expansion is therefore represented, in this case, by $e(V_1-V)$; therefore, $v_1=V-e(V_1-V)=(1+e)V-eV_1$. In like manner, the velocity gained by W_2 during compression is in all cases represented by $(V \mp V_1)$: that gained during expansion is therefore represented by $e(V \mp V_2)$; therefore, $v_2=V+e(V \mp V_2)=(1+e)V \mp eV_2$. Substituting for V, and reducing,

$$v_1=\frac{(W_1-eW_2)V_1 \pm (1+e)W_2V_2}{W_1+W_2} \quad \ldots . (712);$$

$$v_2=\frac{\mp (W_1-W_2)eV_1+(1+e)W_1V_1}{W_1+W_2} \quad \ldots . (713).$$

442. In the impact of two elastic bodies, to determine the accumulated work, or one half the vis viva, lost by the one and gained by the other.

The vis viva lost by W_1 during the impact is evidently represented by $\frac{W_1}{g}V_1^2-\frac{W_1}{g}v_1^2=\frac{W_1}{g}(V_1^2-v_1^2)=\frac{W_1}{g}\left\{V_1^2-\{(1+e)V-eV_1\}^2\right\}=\frac{W_1}{g}\{(1-e^2)V_1^2+2e(1+e)VV_1-(1+e)^2V^2\}=\frac{W_1}{g}(1+e)(V_1-V)\{V_1(1-e)+V(1+e)\}$.

Substituting in this expression its value for V (equation 706), reducing, and representing by u_1 one half the vis viva lost by W_1 in its impact, or the amount by which its accumulated work is diminished by the impact (Art. 67.),

$$u_1=\frac{(1+e)W_1W_2(V_1 \mp V_2)}{2g(W_1+W_2)^2}\{2W_1V_1+(1-e)W_2V_1 \pm (1+e)W_2V_2\} \ldots . (714).$$

Similarly, if u_2 be taken to represent one half the vis viva gained by W_2, or the amount by which its accumulated work is increased by the impact, then

$$u_2=\pm\frac{(1+e)W_1W_2(V_1 \mp V_2)}{2g(W_1+W_2)^2}\{2W_2V_2+(1-e)W_1V_2 \pm (1+e)W_1V_1\} \ldots . (715).$$

445. As an illustration of the principle established in the last article, let it be required to determine the space through which a nail will be driven by the blow of a hammer; and let it be supposed that the resistance opposed to the driving of the nail is partly a constant resistance overcome at its point, and partly a resistance opposed by the friction of the mass into which it is driven upon its sides, varying in amount directly with the length of it x, at any time imbedded in the wood. Let this resistance be represented by $\alpha+\beta x$; then will the work which must be expended in driving it to a depth D be represented (Art. 51.) by

$$\int_0^D(\alpha+\beta x)dx, \text{ or by } (\alpha D+\tfrac{1}{2}\beta D^2).$$

Let W_2 represent the weight of the nail, and V the velocity with which a hammer whose weight is W_1 must impinge upon it to drive it to this depth, and let the surfaces of the nail and hammer both be supposed inelastic; then will the work

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$\frac{W_2}{g}f_2$ represent the effective forces upon the two bodies at any period of the impact; then, by D'Alembert's principle,

$$\frac{W_1}{g}f_1-P_1-\frac{W_2}{g}f_2-P_2=0;$$

or representing by t the time occupied in the impact, up to the period of greatest compression, by V their common velocity at that period, and by v_1 and v_2 their velocities at any period of the impact, and substituting for f_1 and f_2 their values (equation 72),

$$\frac{W_1}{g}\frac{dv_1}{dt}-P_1-\frac{W_2}{g}\frac{dv_2}{dt}-P_2=0.$$

Transposing and integrating between the limits 0 and t,

$$\frac{W_1}{g}(V_1-V)=\frac{W_2}{g}(V-V_1)+\int_0^t(P_1+P_2)dt.$$

Now if P_1 and P_2 be not exceedingly great, the integral in the second member of the equation is exceedingly small as compared with its other terms, and may be neglected; the above equation will then become identical with equation (705).

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$$\tfrac{1}{2}\left(\frac{K_1E_1l_1^2}{L_1}+\frac{K_2E_2l_2^2}{L_2}\right).$$

But this work has been done by the work $\frac{1}{2}\frac{W}{g}V^2$, accumulated (Art. 66.) before impact in the impinging body, and that work has been exhausted in doing it;

$$\therefore\ \tfrac{1}{2}\left(\frac{K_1E_1l_1^2}{L_1}+\frac{K_2E_2l_2^2}{L_2}\right)=\tfrac{1}{2}\frac{W}{g}V^2.$$

Moreover, the mutual pressures upon the surfaces of contact are at every period of the impact equal, and at the instant of greatest compression they are represented respectively (equation 490) by $\frac{K_1E_1l_1}{L_1}$ and $\frac{K_2E_2l_2}{L_2}$;

$$\therefore\ \frac{K_1E_1l_1}{L_1}=\frac{K_2E_2l_2}{L_2}=P \quad \ldots\ldots (718).$$

Eliminating l_2 between this equation and the preceding, and reducing,

$$l_1=\frac{L_1V}{K_1E}\left\{\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)g\right\}^{-\frac{1}{2}}\sqrt{W} \quad \ldots\ldots (719);$$

$$P=V\left\{\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)g\right\}^{-\frac{1}{2}}\sqrt{W} \quad \ldots\ldots (720);$$

in which expressions l_1 represents the greatest compression of the prism whose section is K_1, and P the driving pressure at the instant of greatest compression.

447. *The mutual pressures* P *of the surfaces of contact at any period of the impact.*

Let l represent the space described by that extremity of the impinging prism, by which it does not impinge: it is evident that this space is made up of the two corresponding compressions of the surfaces of impact of the prisms; so that if these be represented by l_1 and l_2, then $l=l_1+l_2$. But

(equation 718) $l_1=\frac{PL_1}{K_1E_1}$, $l_2=\frac{PL_2}{K_2E_2}$; therefore $l=P\left(\frac{L_1}{K_1E}+\frac{L_2}{K_2E_2}\right)$;

$$\therefore\ P=l\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)^{-1} \quad \ldots \quad (721).$$

448. *A measure of the compressibility of the prisms.*

If λ be taken to represent the space through which that extremity of the impinging prism by which it does *not* impinge will have moved when the mutual pressure of the surfaces of contact is 1 lb.; or, in other words, if λ represent the aggregate space through which the prisms would be compressed by a pressure of 1 lb.; then, by the preceding equation,

$$\lambda=\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2} \quad \ldots \quad (722).$$

λ may be taken as *a measure of the aggregate compressibility of the prisms, being the space through which their opposite extremities would be made to approach one another by a pressure of* 1 *lb. applied in the direction of their length.*

If λ_1 and λ_2 represent the spaces through which the prisms would *severally* be compressed by pressures of 1 lb. applied to each, then $\lambda_1=\frac{L_1}{K_1E_1}$, $\lambda_2=\frac{L_2}{K_2E_2}$; therefore $\lambda=\lambda_1+\lambda_2$, or the aggregate compressibility of the two prisms is equal to the sum of their separate compressibilities.

449. *The work* U *expended upon the compression of the prisms at any period of the impact.*

The work expended upon the compression l_1 is represented by $\frac{1}{2}\frac{K_1E_1}{L_1}l_1^2$; or, substituting its value for l_1 (equation 718), it is represented by $\frac{1}{2}\frac{L_1}{K_1E_1}P^2$. And, similarly, the work

expended on the compression l_2 is represented by $\frac{1}{2}\frac{L_2}{K_2E_2}P^2$; therefore $u=\frac{1}{2}\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)P^2$; or substituting for P its value from equation (721),

$$u=\frac{1}{2}l^2\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)^{-1}=\frac{1}{2}\frac{l^2}{\lambda} \quad . \quad . \quad . \quad . \quad (723).$$

450. *The velocity of the impinging body at any period of the impact, the impact being supposed to take place vertically.*

It is evident that at any period of the impact, when the velocity of the impinging body is represented by v, there will have been expended, upon the compression of the two bodies, an amount of work which is represented by the work accumulated in the impinging body before impact, increased by the work done upon it by gravity during the impact, and diminished by that which still remains accumulated in it, or by $\frac{1}{2}\frac{W}{g}V^2+Wl-\frac{1}{2}\frac{W}{g}v^2$.

Representing, therefore, by u the work expended upon the compression of the bodies, we have $\frac{1}{2}\frac{W}{g}V^2+Wl-\frac{1}{2}\frac{W}{g}v^2=u$.

Substituting, therefore, for u its value from equation (723),

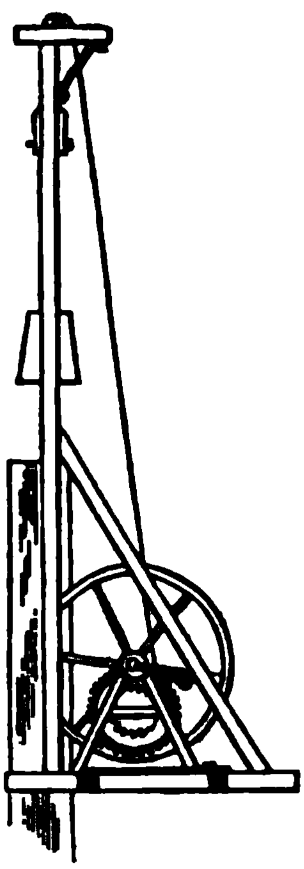
$$\frac{1}{2}\frac{W}{g}V^2+Wl-\frac{1}{2}\frac{W}{g}v^2=\frac{1}{2}l^2\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)^{-1};$$

$$\therefore \quad v^2=V^2+2lg-\frac{l^2g}{W}\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)^{-1} \quad . \quad . \quad . \quad . \quad (724).$$

Or substituting for l its value in terms of P (equation 721),

$$v^2=V^2+g\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)\left(2P-\frac{P^2}{W}\right) \quad . \quad . \quad . \quad . \quad (725).$$

The Pile Driver.



451. It is evident that the pile will not begin to be driven until a period of the impact is attained, when the pressure of the ram upon its head, together with the weight of the pile, exceeds the resistance opposed to its motion by the coherence and the friction of the mass into which it is driven. Let this resistance be represented by P; let V represent the velocity of the ram at the instant of impact, and v its velocity at the instant when the pile begins to move, and W_1, W_2 the weights of the ram and pile; then, since the pile will have been at rest during the whole of the intervening period of the impact, since moreover the mutual pressures Q of the surfaces of contact are, at the instant of motion, represented by $P-W_2$, we have by equation (725)

$$v^2=V^2-g\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)\left\{\frac{(P-W_2)^2}{W_1}-2(P-W_2)\right\} \quad . \quad . \quad . \quad (726).$$

If the value of v determined by this equation be not a possible quantity, no motion can be communicated to the pile by the impact of the ram: the following inequality is therefore a condition necessary to the driving of the pile,

$$V^2>g\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)\left\{\frac{(P-W_2)^2}{W_1}-2(P-W_2)\right\} \quad . \quad . \quad (727).$$

After the pile has moved through any given distance, one portion of the work accumulated in the ram before its impact will have been expended in overcoming, through that distance, the resistance opposed to the motion of the pile; another portion will have been expended upon the compression of the surfaces of the ram and pile; and the remainder will be accumulated in the moving masses of the ram and pile. The motion of the pile cannot cease until after the period of the greatest compression of the ram and pile is attained; since the

impinging prism at the instant of impact; and let x_1 represent the space through which the aggregate length BP of the two prisms has been diminished since that period of the impact, and x_2 the space through which the point P has moved; then (equation 721)

$$Q=x_1\left(\frac{L_1}{K_1E_1}+\frac{L_2}{K_2E_2}\right)^{-1}=\frac{x_1}{\lambda} \quad \ldots \quad (730).$$

Also $AB=x_1+x_2$; therefore velocity of point $B=\frac{d(x_1+x_2)}{dt}$, (Art. 96.); therefore $f_1=\frac{d^2x_1}{dt^2}+\frac{d^2x_2}{dt^2}=\frac{d^2x_1}{dt^2}+f_2$.

Substituting these values of f_1 and Q in equations (728) and (729), and eliminating f_2 between the resulting equations,

$$\frac{d^2x_1}{dt^2}=-\frac{g}{\lambda}\left(\frac{1}{W_1}+\frac{1}{W_2}\right)x_1+\frac{Pg}{W_2} \quad \ldots \ldots \quad (731).$$

Integrating this equation by the known rules, we obtain

$$x_1=A \sin. \gamma t+B \cos. \gamma t+\frac{Pg}{\gamma^2 W_2} \quad \ldots \ldots \quad (732);$$

in which expression the value of γ is determined by the equation

$$\gamma^2=\frac{g}{\lambda}\left(\frac{1}{W_1}+\frac{1}{W_2}\right)=g\left\{\frac{W_1^{-1}+W_2^{-1}}{L_1(K_1E_1)^{-1}+L_2(K_2E_2)^{-1}}\right\} \quad . . \quad (733);$$

and A and B are certain constants to be determined by the conditions of the question. Substituting in equation (729) the value of Q from equation (730), and solving in respect to f_2,

$$f_2=\frac{g}{W_2\lambda}x_1+\left(1-\frac{P}{W_2}\right)g \quad \ldots \ldots \quad (734).$$

Substituting for x_1 its value from equation (732), and for f_2 its value $\frac{d^2x_2}{dt^2}$, and reducing,

$$\frac{d^2x_2}{dt^2}=\frac{Ag}{W_2\lambda}\sin. \gamma t+\frac{Bg}{W_2\lambda}\cos. \gamma t+\left(1-\frac{P}{W_1+W_2}\right)g.$$

Integrating between the limits 0 and t, and observing that

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pression of the two prisms, and is represented by $\frac{dx_1}{dt}$. The value of $\frac{dx_1}{dt}$, when $t=0$, is represented therefore by v (equation 726). Differentiating, therefore, equation (732), assuming $t=0$, and substituting v for $\frac{dx_1}{dt}$, we obtain $v=\gamma A$; whence it appears that the value of A is determined by dividing the square root of the second member of equation (726) by γ.

Substituting for A and B their values in equation (736),

$$-\cos.\gamma T)+\lambda W_2\left(\frac{P}{W_1+W_2}-1\right)\sin.\gamma T+\left(1-\frac{P}{W_1+W_2}\right)W_2\lambda\gamma T=0.$$

Reducing, and dividing by the common factor of the two last terms,

$$\frac{v(1-\cos.\gamma T)}{g\lambda W_2\{P(W_1+W_2)^{-1}-1\}}+\sin.\gamma T-\gamma T=0 \;.\;.\;.\;. \quad (739).$$

Substituting for A and B their values in equation (735), and representing by D the value of x_2, when $t=T$,

$$D=\frac{vg}{W_2\lambda\gamma^3}(\gamma T-\sin.\gamma T)+g\left(\frac{P}{W_1+W_2}-1\right)\left(\frac{\text{vers.}\,\gamma T}{\gamma^2}-\tfrac{1}{2}T^2\right) \;.\;.\;.\;. \quad (740).$$

The value of T determined by equation (739) being substituted in equation (740), an expression is obtained for the whole space through which the second prism is driven by the impact of the first.*

* The method of the above investigation is, from equation (731), nearly the same with that given by Dr. Whewell, in the last edition of his Mechanics, the principle of the investigation appears to be due to Mr. Airey. If the value of γ, as determined by equation (739), were not exceedingly great, then, since the value of T is in all practical cases exceedingly small, the value of γT would in all cases be exceedingly small, and we might approximate to the value of T in equation (740), by substituting for cos. γT and sin. γT, the two first terms of the expansions of those functions, in terms of γT.

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APPENDIX.

NOTE A.

THEOREM. — *The definite integral* $\int_a^b fx\,dx$ *is the limit of the sums of the values severally assumed by the product* $fx \,.\, \Delta x$, *as* x *is made to vary by successive equal increments of* Δx, *from* a *to* b, *and as each such equal increment is continually and infinitely diminished, and their number therefore continually and infinitely increased.*

To prove this, let the general integral be represented by Fx; let us suppose that fx does not become infinite for any value of x between a and b, and let any two such values be x and $x+\Delta x$; therefore, by Taylor's theorem, $F(x+\Delta x)=Fx+\Delta x fx+(\Delta x)^{1+\lambda}M$, where the exponent $1+\lambda$ is given to the third term of the expansion instead of the exponent 2, that the case may be included in which the second differential coefficient of Fx, $\frac{dfx}{dx}$, is infinite, and in which the exponent of Δx in that term is therefore a fraction less than 2.

Let the difference between a and b be divided into n equal parts; and let each be represented by Δx, so that

$$\frac{b-a}{n}=\Delta x.$$

Giving to x, then, the successive values a, $a+\Delta x$, $a+2\Delta x \,..\, a+(n-1)\Delta x$, and adding,

$$F(a+n\Delta x)=Fa+\Delta x\Sigma_1^n f\{a+(n-1)\Delta x\}+(\Delta x)^{1+\lambda}\Sigma M_n,$$

$$\therefore\ Fb-Fa=\Delta x\Sigma_1^n f\{a+(n-1)\Delta x\}+(\Delta x)^{1+\lambda}\Sigma M_n.$$

Now none of the values of M are infinite, since for none of these values is fx infinite. If, therefore, M be the greatest of these values, then is ΣM_n less than nM: and therefore

$$Fb-Fa-\Delta x\Sigma_1^n f\{a+(n-1)\Delta x\}<(b-a)M(\Delta x)^{\lambda}.$$

The difference of the definite integral $Fb-Fa$, and the sum $\Sigma_1^n(\Delta x)f\{a+(n-1)\Delta x\}$ is always, therefore, less than $(b-a)M(\Delta x)^{\lambda}$. Now M is *finite*, and $(b-a)$ is *given*, and as n is increased Δx is diminished continually; and therefore $(\Delta x)^{\lambda}$ is diminished continually, λ being positive

Thus by increasing n indefinitely, the difference of the definite integ

than cot. ψ_2. Now, so long as all the values of the error (formula 1) remain positive, between the proposed limits, they are all manifestly diminished by diminishing α and β; but when by this diminution the error is at length rendered negative in respect to one or both of the extreme values ψ_1, or ψ_2 of ψ, and to others adjacent to them, then do these negative errors continually *increase*, as α and β are yet farther diminished, whilst the positive maximum error (formula 3) continually *diminishes*. Now the most favourable condition, in respect to the whole range of the errors between the proposed limits of variation, will manifestly be attained when, by thus diminishing the positive and thereby increasing the negative errors, the greatest positive error is rendered equal to each of the two negative errors; a condition which will be found to be consistent with that before made in respect to the arbitrary values of α and β, and which supposes that the values of the error (formula 1) corresponding to the values ψ_1 and ψ_2 are each equal, when taken negatively, to the maximum error represented by formula 3, or that the constants α and β are taken so as to satisfy the two following equations.

$$1-(\alpha\cos.\psi_1+\beta\sin.\psi_1)=\sqrt{\alpha^2+\beta^2}-1.$$

$$1-(\alpha\cos.\psi_1+\beta\sin.\psi_1)=1-(\alpha\cos.\psi_2-\beta\sin.\psi_2).$$

The last equation gives us by reduction

$$\alpha\cos.\psi_1+\beta\sin.\psi_1=\beta\,\frac{\cos.\frac{1}{2}(\psi_1-\psi_2)}{\sin.\frac{1}{2}(\psi_1+\psi_2)},$$

$$\text{and } \alpha=\beta\cot.\tfrac{1}{2}(\psi_1+\psi_2).$$

Substituting these values in the first equation, and reducing,

$$\beta=\frac{2\sin.\frac{1}{2}(\psi_1+\psi_2)}{1+\cos.\frac{1}{2}(\psi_1-\psi_2)}=\frac{\sin.\frac{1}{2}(\psi_1+\psi_2)}{\cos.^2\frac{1}{4}(\psi_1-\psi_2)}\quad\ldots\ldots\ (4);$$

$$\therefore\ \alpha=\frac{2\cos.\frac{1}{2}(\psi_1+\psi_2)}{1+\cos.\frac{1}{2}(\psi_1-\psi_2)}=\frac{\cos.\frac{1}{2}(\psi_1+\psi_2)}{\cos.^2\frac{1}{4}(\psi_1-\psi_2)}\quad\ldots\ldots\ (5).$$

These values of α and β give for the maximum error (formula 3) the expression

$$\tan.^2\tfrac{1}{4}(\psi_1-\psi_2)\quad\ldots\ldots\ (6).$$

Thus, then, it appears that the value of the radical $\sqrt{a^2+b^2}$ is represented, in respect to all those values of $\frac{a}{b}$ which are included between the limits cot. ψ_1 and cot. ψ_2, by the formula

$$\frac{\cos.\frac{1}{2}(\psi_1+\psi_2)}{\cos.^2\frac{1}{4}(\psi_1-\psi_2)}+b\,\frac{\sin.\frac{1}{2}(\psi_1+\psi_2)}{\cos.^2\frac{1}{4}(\psi_1-\psi_2)}\quad\ldots\ldots\ (7),$$

with a degree of approximation which is determined by the value of $\tan.^2\frac{1}{4}(\psi_1-\psi_2)$.

If in the proposed radical the value of a admits of being increased infinitely in respect to b, or the value of b infinitely diminished in respect to a, then cot. ψ_1= infinity; therefore $\psi_1=0$. In this case the formula of approximation becomes

[redacted]

PONCELET'S SECOND THEOREM.

To approximate to the value of $\sqrt{a^2-b^2}$, let $\alpha a-\beta b$ be the formula of approximation, then will the relative error be represented by

$$\frac{\sqrt{a^2-b^2}-(\alpha a-\beta b)}{\sqrt{a^2-b^2}}, \text{ or by } 1-\frac{\left(\alpha\frac{a}{b}-\beta\right)}{\sqrt{\frac{a^2}{b^2}-1}}.$$

Now, let it be observed that a^2 being essentially greater than b^2, $\frac{a}{b}>1$; let $\frac{a}{b}$, therefore, be represented by cosec. ψ, then will the relative error be represented by $1-\frac{(\alpha \operatorname{cosec.}\psi-\beta)}{\sqrt{\operatorname{cosec.}^2\psi-1}}$, or by

$$1-\alpha \sec.\psi+\beta \tan.\psi \;.\,.\,.\,.\,.\; (12),$$

which function attains its maximum when $\sin.\psi=\frac{\beta}{\alpha}$. Substituting this value in the preceding formula, and observing that $-\alpha \sec.\psi+\beta\tan.\psi=$ $-\sec.\psi(\alpha-\beta\sin.\psi)=-\frac{\left(\alpha-\frac{\beta^2}{\alpha}\right)}{\sqrt{1-\frac{\beta^2}{\alpha^2}}}=-\sqrt{\alpha^2-\beta^2}$, we obtain for the maximum error the expression

$$1-\sqrt{\alpha^2-\beta^2} \;.\,.\,.\,.\,.\; (13).$$

Assuming ψ_1 and ψ_2 to represent the values of ψ, corresponding to the greatest and least values of $\frac{a}{b}$, and observing that in this case, as in the preceding, the values of α and β, which satisfy the conditions of the question, are those which render the values of the error corresponding to these limits equal, when taken with contrary signs, to the maximum error, we have

$$-1+\alpha\sec.\psi_1-\beta\tan.\psi_1=1-\sqrt{\alpha^2-\beta^2} \;.\,.\,.\,.\; (14).$$

$$1-\alpha\sec.\psi_1+\beta\tan.\psi_1=1-\alpha\sec.\psi_2+\beta\tan.\psi_2 \;.\,.\,.\,.\; (15).$$

The latter equation gives, by reduction,

$$\alpha=\beta\frac{\cos.\frac{1}{2}(\psi_1-\psi_2)}{\sin.\frac{1}{2}(\psi_1+\psi_2)} \;.\,.\,.\,.\; (16).$$

$$\alpha^2-\beta^2=\beta^2\left\{\frac{\cos.^2\frac{1}{2}(\psi_1-\psi_2)}{\sin.^2\frac{1}{2}(\psi_1+\psi_2)}-1\right\}=\beta^2\frac{\cos.\psi_1\cos.\psi_2}{\sin.^2\frac{1}{2}(\psi_1+\psi_2)}.$$

And $\alpha\sec.\psi_1-\beta\tan.\psi_1=\beta\cot.\frac{1}{2}(\psi_1+\psi_2) \;.\,.\,.\,.\; (17).$

Substituting these values in equation (14), and solving in respect to β,

NOTE C.

The following is the general theorem expressing the relation between any number of pressures P_1, P_2, P_3, &c., applied to a body moveable about a cylindrical axis, in its state bordering upon motion. It is demonstrated in a "Memoir upon the Theory of Machines," printed in the Second Part of the *Transactions of the Royal Society* for 1841. — Let ι_{12}, ι_{13}, ι_{14}, ι_{23}, &c. be taken to represent the inclinations of the directions of the pressures to one another; a_1, a_2, a_3, &c. the perpendiculars upon them, severally, from the centre of the axis; ρ the radius of the axis; and ϕ the limiting angle of resistance; then

$$P_1 = -\frac{P_2a_2+P_3a_3+\ldots\ldots}{a_1}+\frac{\rho\sin.\phi}{a_1^2}\left\{\begin{matrix}P_2^2L_{12}^2+P_3^2L_{13}^2+P_4^2L_{14}^2+\ldots\ldots\\+2P_2P_3M_{23}+2P_2P_4M_{24}+\ldots\ldots\end{matrix}\right\}^{\frac{1}{2}};$$

in which expression L_{12}, L_{13}, &c. are taken to represent the lines which join the foot of the perpendicular a_1 let fall upon the pressure P_1, with the feet of the perpendiculars a_2, a_3, a_4, &c. let fall upon the other pressures of the system; and in which M_{23}, M_{24}, &c. are taken to represent the different values assumed by the function

$$\{a_2a_3-a_1(a_1\cos.\iota_{23}+a_2\cos.\iota_{13}+a_3\cos.\iota_{12})\}.$$

It is evident that equation (161) is but a particular case of this more general theorem.

NOTE D.

The best Dimensions of a Buttress.

If m_1 (Art. 301.) represent the modulus of stability of the portion AG of the wall, it may be shown, as before, that

$$P\{(h_1-h_2)\sin.a-(l-a_2-m_1)\cos.a\}=(\tfrac{1}{2}a_1-m_1)(h_1-h_2)a_1\mu;$$

$$P\{(h_1-h_2)\sin.a-(l-a_2)\cos.a\}=\tfrac{1}{2}(h_1-h_2)a_1^2\mu-m_1\{P\cos.a+(h_1-h_2)a_1\mu\}\ldots\ldots(25).$$

If $m_1=m$, the stability of the *portion* AG of the structure is the same with that of the *whole* AC; an arrangement by which the greatest strength is obtained with a given quantity of material (see Art. 390.). This supposition being made, and m eliminated between the above equation and equation (393), that relation between the dimensions of the buttress and those of the wall which is consistent with the greatest economy of the material used will be determined. The following is that relation: —

$$\frac{\ldots{}^2h_1+2a_1a_2h_1+\frac{1}{2}a_2^2h_2)-P(h_1\sin.a-l\cos.a)}{P\cos.a+\mu\left(a_1h_1+\frac{1}{2}a_2h_2\right)}=\frac{\frac{1}{2}\mu(h_1-h_2)a_1^2-P\{(h_1-h_2)\sin.a-(l-a_2)\cos.a\}}{P\cos.a+\mu a_1(h_1-h_2)}\ldots(26).$$

Material.	Value of c (equation 238).		Value of C (equation 239).	
	For Teeth of the best Workmanship.	For Teeth of inferior Workmanship.	For Teeth of the best Workmanship.	For Teeth of inferior Workmanship.
Cast iron . -	·004795	·004870	0·912	0·922
Brass - -	·005982	·006077	1·057	1·068
Hard wood -	·006621	·006726	1·131	1·143

The following are the pitches commonly in use among mechanics: —

in. in. in. in. in. in. in.
$1, \; 1\frac{1}{8}, \; 1\frac{1}{4}, \; 1\frac{1}{2}, \; 2, \; 2\frac{1}{2}, \; 3.$

Prof. Willis considers the following to be sufficient below inch pitch: —

in. in. in. in. in.
$\frac{1}{4}, \; \frac{3}{8}, \; \frac{1}{2}, \; \frac{5}{8}, \; \frac{3}{4}.$

Having, therefore, determined the proper pitch to be given to the tooth from formula 239, the nearest pitch is to be taken from the above series to that thus determined.

NOTE F.

Experiments of M. Morin on the Traction of Carriages.

The following are among the general results deduced by M. Morin from his experiments: —

1. The traction is directly proportional to the load, and inversely proportional to the diameter of the wheel.
2. Upon a paved or a hard Macadamized road, the resistance is independent of the width of the tire when it exceeds from 3 to 4 inches.
3. At a walking pace the traction is the same, under the same circumstances, for carriages with springs and without them.
4. Upon hard Macadamized and upon paved roads, the traction increases with the velocity; the increments of traction being directly proportional to the increments of velocity above the velocity 3·28 feet per second, or about $2\frac{1}{4}$ miles per hour. The equal increment of traction thus due to each equal increment of velocity is less as the road is more smooth, and the carriage less rigid or better hung.
4. Upon soft roads of earth, or sand or turf, or roads fresh and thickly gravelled, the traction is independent of the velocity.
5. Upon a well-made and compact pavement of hewn stones, the traction at a walking pace is not more than three fourths of that upon the best Macadamized road under similar circumstances; at a trotting pace it is equal to it.
6. The destruction of the road is in all cases greater as the diameters of the wheels are less, and it is greater in carriages without than with springs.

NOTE G.

On the Strength of Columns.

Mr. Hodgkinson has obligingly communicated the following observations on Art. 432.: —

1. The reader must be made to understand that the rounding of the ends of the pillars is to make them moveable there, as if they turned by means of an universal joint; and the flat-ended pillars are conceived to be supported in every part of the ends by means of flat surfaces or otherwise, rendering the ends perfectly immoveable.

2. The coefficient (13) for hollow columns with rounded ends is deduced from the whole of the experiments first made, including some which were very defective on account of the difficulty experienced in the earlier attempts to cast good hollow columns so small as were wanted. The first castings were made lying on their side; and this, notwithstanding every effort, prevented the core being in the middle: some of the columns were reduced, too, in thickness, half way between the middle and the ends, and near to the ends, and this slightly reduced the strength. These causes of weakness existed much more among the pillars with rounded ends than those with flat ones; they are alluded to in the paper (Art. 47.). Had it not been for them, the coefficient (13) would, I conceive, have been equal to that for solid pillars (or 14·9).

3. The fact of long pillars with flat ends being about three times as strong as those of the same dimensions with rounded ends is, I conceive, well made out, in cast iron, wrought iron, and timber; you have, however, omitted it, being perhaps led to do it through the low value of the coefficient (13) above mentioned.

The same may be mentioned with respect to the near approach in strength of long pillars with flat ends, and those of half the length with rounded ends. It may be said that the law of the 1·7 power of the length would nearly indicate the latter; but this last, and the other powers 3·76 and 3·55, are only approximations, and not exactly constant, though nearly so, and I do not know whether the other equal quantities are not, with some slight modifications, physical facts.

4. The strength of pillars of *similar form* and of the same materials varies as the 1·865 power, or nearly as the square of their like linear dimensions, or as the area of their cross section.

TABLE I.

The Numerical Values of COMPLETE *Elliptic Functions of the* FIRST *and* SECOND *Orders for Values of the Modulus* k *corresponding to each Degree of the Angle* sin.^{-1}k.

Sin.^{-1}k.	F_1.	E_1.	Sin.^{-1}k.	F_1.	E_1.
0°	1·57079	1·57079	46	1·86914	1·34180
1	1·57091	1·57067	47	1·88480	1·33286
2	1·57127	1·57031	[illegible]	1·90108	1·32384
3	1·57187	1·56972	49	1·91799	1·31472
4	1·57271	1·56888	50	1·93558	1·30553
5	1·57379	1·56780	[illegible]	1·95386	1·29627
6	1·57511	1·56649	52	1·97288	1·28695
7	1·57667	1·56494	53	1·99266	1·27757
8	1·57848	1·56316	54	2·01326	1·26814
9	1·58054	1·56114	55	2·03471	1·25867
10	1·58284	1·55888	56	2·05706	1·24918
11	1·58539	1·55639	57	2·08035	1·23966
12	1·58819	1·55368	58	2·10465	1·23012
13	1·59125	1·55073	59	2·13002	1·22058
14	1 59456	1·54755	60	2·15651	1·21105
15	1·59814	1·54415	61	2·18421	1·20153
16	1·60197	1·54052	62	2·21319	1·19204
17	1·60608	1·53666	63	2·24354	1·18258
18	1·61045	1·53259	64	2·27537	1·17317
19	1·61510	1·52830	65	2·30878	1·16382
20	1·62002	1·52379	66	2·34390	1·15454
21	1·62523	1·51907	67	2·38087	1·14534
22	1·63072	1·51414	68	2·41984	1·13624
23	1·63651	1·50900	69	2·46099	1·12724
24	1·64260	1·50366	70	2·50455	1·11837
25	1·64899	1·49811	71	2 55073	1·10964
26	1·65569	1·49236	72	2·59981	1·10106
27	1·66271	1·48642	[illegible]	2·65213	1·09265
28	1·67006	1·48029	74	2·70806	1·08442
29	1·67773	1·47396	75	2·76806	1·07640
30	1·68575	1·46746	76	2·83267	1·06860
31	1·69411	1·46077	77	2·90256	1·06105
32	1·70283	1·45390	78	2·97856	1·05377
33	1·71192	1·44686	79	3·06172	1·04678
34	1·72139	1·43966	80	3·15338	1·04011
35	1·73124	1·43229	81	3·25530	1·03378
36	1·74149	1·42476	82	3·36986	1·02784
37	1·75216	1·41707	83	3·50042	1·02231
38	1·76325	1·40923	84	3·65185	1·01723
39	1·77478	1·40125	85	3·83174	1·01266
40	1·78676	1·39314	86	4·05275	1·00864
41	1·79922	1·38488	87	4·33865	1·00525
[illegible]	1·81215	1 37650	88	4·74271	1·00258
43	1·82560	1·36799	[illegible]	5·43490	1·00075
44	1·83956	1·35937			
45	1·85407	1·35064			

[redacted]

$\iota=15°$.

a	c=0	c=0·1	c=0·2	c=0·3	c=0·4	c=0·5	c=1·0
0·05	64·8°	50·5°	46·95°	45·69°	45·03°	44·67°	43·9°
0·10	59·3	55·07	53·34	52·47	51·99	51·69	50·93
0·15	59·08	57·32	56.65	56·05	55·75	55·55	55·05
0·20	59·06	58·60	58·35	58·20	58·10	58·02	57·84
0·25	59·05	59·28	59·42	59·53	59·60	59·65	59·79
0·30	58·90	59·57	59·98	60·26	60·48	60·66	61·15
0·35	58·53	59·41	60·09	60·57	60·93	61·17	62·0
0·40	57·99	59·08	59·87	60·48	60·95	61·36	62·6
0·45	57·26	58·43	59·34	60·06	60·67	61·15	62·7
0·50	56·38	57·61	58·58	59·36	60·06	60·64	62·5

$\iota=22°\ 30'$.

a	c=0	c=0·1	c=0·2	c=0·3	c=0·4	c=0·5	c=1·0
0·05	36·1°	41·2°	42·0°	42·3°	42·6°	42·7°	42·9°
0·10	50·5	50·3	50·19	50·17	50·14	50·13	50·11
0·15	54·25	54·31	54·35	54·35	54·36	54·36	54·38
0·20	56·17	56·60	56·82	56·95	57·04	57·11	57·28
0·25	57·27	57·93	58·33	58·61	58·79	58·95	59·33
0·30	57·85	58·68	59·23	59·60	59·93	60·16	60·83
0·35	58·07	59·01	59·70	60·21	60·61	60·91	61·85
0·40	58·02	59·02	59·79	60·38	60·87	61·25	62·2
0·45	57·74	58·78	59·60	60.26	60·82	61·27	62·7
0·50	57·30	58·31	59·16	59·88	60·47	61·00	62·9

$\iota=30°$.

a	c=0	c=0·1	c=0·2	c=0·3	c=0·4	c=0·5	c=1·0
0·05	31·3°	36·2°	38·4°	39·57°	40·28°	40·77°	41·9°
0·10	43·3	46·06	47·25	47·90	48·30	48·59	49·24
0·15	50·07	51·46	52·18	52·63	52·94	53·14	53·68
0·20	53·66	54·69	55·27	55·67	55·96	56·16	56·72
0·25	55·80	56·72	57·30	57·72	58·01	58·23	58·89
0·30	57·13	58·01	58·62	59·06	59·40	59·69	60·48
0·35	57·93	58·80	59·43	59·94	60·33	60·66	61·64
0·40	58·33	59·20	59·89	60·42	60·87	61·23	62·39
0·45	58.47	59·33	60·03	60·61	61·08	61·48	62·87
0·50	58·38	59·22	59·93	60·53	61·03	61·47	63·0

$\iota = 37^\circ\ 30'$.

a	$c=0$	$c=0·1$	$c=0·2$	$c=0·3$	$c=0·4$	$c=0·5$	$c=1·0$
0·05	31·1°	34·3°	36·28°	37·59°	38·48°	39·16°	40·82°
0·10	40·98	43·59	45·09	46·01	46·67	47·14	48·35
0·15	47·71	49·40	50·43	51·12	51·61	51·96	52·93
0·20	52·01	53·23	54·01	54·54	54·94	55·24	56·10
0·25	54·87	55·80	56·45	56·94	57·29	57·59	58·41
0·30	56·77	57·58	58·16	58·62	58·98	59·26	60·16
0·35	58·04	58·78	59·34	59·81	60·17	60.47	61·45
0·40	58·89	59·58	60·13	60·60	60·97	61·30	62·4
0·45	59·38	60·06	60·62	61·07	61·47	61·83	63·0
0·50	56·69	60·29	60·84	61·26	61·72	62·07	63·3

$\iota = 45^\circ$.

a	$c=0$	$c=0·1$	$c=0·2$	$c=0·3$	$c=0·4$	$c=0·5$	$c=1·0$
0·05	31·3°	33·68°	35·46°	36·36°	37·22°	38·0°	39·9°
0·10	40·6	42·4	43·7	44·64	45·35	45·92	47·45
0·15	46·77	48·20	49·18	49·93	50·47	50·92	52·15
0·20	51·23	52·27	53·05	53·64	54·07	54·42	55·47
0·25	54·42	55·22	55·84	56·31	56·70	57·01	57·97
0·30	56·72	57·38	57·90	58·30	58·65	58·94	59·85
0·35	58·35	58·94	59·40	59·79	60·11	60·38	61·30
0·40	59·56	60·09	60·52	60·89	61·19	61·46	62·4
0·45	60·40	60·89	61·29	61·67	61·97	62·24	63·2
0·50	60·99	61·43	61·8	62·2	62·5	62·8	63·8

The Tables of M. Garidel.

TABLE III.

Showing the Horizontal Thrust of an Arch, the Radius of whose Intrados is Unity, and the Weight of each Cubic Foot of its Material and of that of its Loading, Unity. (See Art. 346.)

N. B. To find the horizontal thrust of any other arch, multiply that given in the table by the square of the radius of the intrados and by the weight of a cubic foot of the material.

$\iota=0.$

a	$c=0$ $\frac{P}{r^2}$	$c=0{\cdot}1$ $\frac{P}{r^2}$	$c=0{\cdot}2$ $\frac{P}{r^2}$	$c=0{\cdot}3$ $\frac{P}{r^2}$	$c=0{\cdot}4$ $\frac{P}{r^2}$	$c=0{\cdot}5$ $\frac{P}{r^2}$	$c=1{\cdot}0$ $\frac{P}{r^2}$
0·05	0·08174	0·14797	0·21762	0·28877	0·36060	0·43277	0·79541
0·10	0·10279	0·16370	0·22588	0·28862	0·35164	0·41481	0·73161
0·15	0·11894	0·17480	0·23111	0·28764	0·34429	0·40100	0·68504
0·20	0·13073	0·18191	0·23322	0·28460	0·33603	0·38747	0·64488
0·25	0·13871	0·18553	0·23237	0·27922	0·32607	0·37293	0·60727
0·30	0·14333	0·18604	0·22874	0·27145	0·31416	0·35687	0·57041
0·35	0·14504	0·18379	0·22258	0·26140	0·30023	0·33907	0·53335
0·40	0·14422	0·17913	0·21415	0·24924	0·28437	0·31953	0·49560
0·45	0·14124	0·17240	0·20374	0·23520	0·26674	0·29835	0·45693
0·50	0·13649	0·16396	0·19168	0·21957	0·24760	0·27573	0·41728

$\iota=7^\circ\ 30'.$

a	$c=0$ $\frac{P}{r^2}$	$c=0{\cdot}1$ $\frac{P}{r^2}$	$c=0{\cdot}2$ $\frac{P}{r^2}$	$c=0{\cdot}3$ $\frac{P}{r^2}$	$c=0{\cdot}4$ $\frac{P}{r^2}$	$c=0{\cdot}5$ $\frac{P}{r^2}$	$c=1{\cdot}0$ $\frac{P}{r^2}$
0·05	0·06180	0·12867	0·19937	0·27125	0·34356	0·41606	0·77944
0·10	0·08514	0·14666	0·20930	0·27237	0·33561	0·39895	0·71618
0·15	0·10380	0·16001	0·21657	0·27326	0·33003	0·38683	0·67110
0·20	0·11813	0·16948	0·22089	0·27237	0·32384	0·37533	0·63286
0·25	0·12870	0·17557	0·22244	0·26932	0·31619	0·36306	0·59743
0·30	0·13598	0·17866	0·22134	0·26403	0·30673	0·34943	0·56295
0·35	0·14040	0·17909	0·21783	0·25661	0·29542	0·33424	0·52846
0·40	0·14234	0·17718	0·21215	0·24720	0·28230	0·31744	0·4[illegible]
0·45	0·14211	0·17323	0·20454	0·23598	0·26751	0·29[illegible]	
0·50	0·14003	0·16753	0·19528	0·22319	0·25194	[illegible]	

$\iota=15°$.

a	$c=0$ $\frac{P}{r^2}$	$c=0·1$ $\frac{P}{r^2}$	$c=0·2$ $\frac{P}{r^2}$	$c=0·3$ $\frac{P}{r^2}$	$c=0·4$ $\frac{P}{r^2}$	$c=0·5$ $\frac{P}{r^2}$	$c=1·0$ $\frac{P}{r^2}$
0·05	0·05310	0·12265	0·19488	0·26748	0·34018	0·41293	0·77681
0·10	0·07903	0·14170	0·20493	0·26832	0·33176	0·39524	0·71277
0·15	0·09990	0·15658	0·21336	0·27022	0·32708	0·38395	0·66840
0·20	0·11631	0·16781	0·21931	0·27083	0·32234	0·37386	0·63145
0·25	0·12894	0·17582	0·22268	0·26955	0·31643	0·36330	0·59767
0·30	0·13835	0·18096	0·23361	0·26627	0·30895	0·35163	0·56510
0·35	0·14494	0·18355	0·22224	0·26098	0·29976	0·33855	0·53271
0·40	0·14905	0·18384	0·21878	0·25380	0·28888	0·32399	0·49995
0·45	0·15097	0·18212	0·21344	0·24488	0·27641	0·30800	0·46652
0·50	0·15099	0·17860	0·20642	0·23439	0·26247	0·29065	0·43232

$\iota=22° 30'$.

a	$c=0$ $\frac{P}{r^2}$	$c=0·1$ $\frac{P}{r^2}$	$c=0·2$ $\frac{P}{r^2}$	$c=0·3$ $\frac{P}{r^2}$	$c=0·4$ $\frac{P}{r^2}$	$c=0·5$ $\frac{P}{r^2}$	$c=1·0$ $\frac{P}{r^2}$
0·05	0·06102	0·13346	0·20621	0·27899	0·35178	0·42458	0·78857
0·10	0·08700	0·15053	0·21407	0·27760	0·34113	0·40466	0·72233
0·15	0·10877	0·16567	0·22257	0·27947	0·33638	0·39328	0·67778
0·20	0·12635	0·17785	0·22936	0·28087	0·33239	0 38391	0·64150
0·25	0·14037	0·18716	0·23399	0·28082	0·32767	0·37453	0·60886
0·30	0·15129	0·19381	0·23640	0·27902	0·32166	0·36432	0·57773
0·35	0·15948	0·19804	0·23669	0·27540	0·31415	0·35292	0·54700
0·40	0·16525	0·20005	0·23497	0·26999	0·30506	0·34017	0·51608
0·45	0·16883	0·20005	0.23141	0·26289	0·29444	0·32604	0·48460
0·50	0·17047	0·19824	0·22617	0·25423	0·28238	0·31060	0·45241

$\iota=30°$.

a	$c=0$ $\frac{P}{r^2}$	$c=0·1$ $\frac{P}{r^2}$	$c=0·2$ $\frac{P}{r^2}$	$c=0·3$ $\frac{P}{r^2}$	$c=0·4$ $\frac{P}{r^2}$	$c=0·5$ $\frac{P}{r^2}$	$c=1·0$ $\frac{P}{r^2}$
0·05	0·09355	0·16408	0·23605	0·30845	0·38101	0·45365	0·81731
0·10	0·11297	0·17592	0·23922	0·30263	0·36609	0·42957	0·74711
0·15	0·13295	0·18962	0·24640	0·30323	0·36009	0·41696	0·70138
0·20	0·15038	0·20172	0·25314	0·30459	0·35606	0·40755	0·66506
0·25	0·16493	0·21160	0·25834	0·30513	0·35193	0·39876	0·63299
0·30	0·17673	0·21917	0·26170	0·30427	0·34688	0·38951	0·60282
0·35	0·18599	0·22452	0·26314	0·30182	0·34055	0·37930	0·57332
0·40	0·19293	0·22777	0·26271	0·29773	0·33280	0·36791	0·54380
0·45	0·19774	0·22906	0·26050	0·29202	0·32361	0·35524	0·51385
0·50	0·20060	0·22854	0·25661	0·28476	0·31299	0·34128	0·48327

$\iota=37^\circ\ 30'$.

a	$c=0$ $\frac{P}{r^2}$	$c=0{\cdot}1$ $\frac{P}{r^2}$	$c=0{\cdot}2$ $\frac{P}{r^2}$	$c=0{\cdot}3$ $\frac{P}{r^2}$	$c=0{\cdot}4$ $\frac{P}{r^2}$	$c=0{\cdot}5$ $\frac{\ }{r^2}$	$c=1{\cdot}0$ $\frac{P}{r^2}$
0·05	0·14749	0·21733	0·28854	0·36038	0·43255	0·50490	0·86784
0·10	0·15949	0·22174	0·28457	0·34768	0·41093	0·47426	0·79141
0·15	0·17605	0·23233	0·28886	0·34553	0·40226	0·45904	0·74322
0·20	0·19209	0·24321	0·29448	0·34583	0·39722	0·44865	0·70598
0·25	0·20627	0·25282	0·29948	0·34619	0·39294	0·43972	0·67382
0·30	0·21827	0·26066	0·30314	0·34568	0·38825	0·43085	0·64406
0·35	0·22805	0·26659	0·30521	0·34388	0·38259	0·42133	0·61529
0·40	0·23570	0·27060	0·30558	0·34062	0·37571	0·41083	0·58673
0·45	0·24130	0·27275	0·30427	0·33586	0·36749	0·39916	0·55787
0·50	0·24499	0·27312	0·30132	0·32958	0·35789	0·38625	0·52845

$\iota=45^\circ$.

a	$c=0$ $\frac{P}{r^2}$	$c=0{\cdot}1$ $\frac{P}{r^2}$	$c=0{\cdot}2$ $\frac{P}{r^2}$	$c=0{\cdot}3$ $\frac{P}{r^2}$	$c=0{\cdot}4$ $\frac{P}{r^2}$	$c=0{\cdot}5$ $\frac{P}{r^2}$	$c=1{\cdot}0$ $\frac{P}{r^2}$
0·05	0·23105	0·30081	0·37162	0·44305	0·51485	0·58688	0·94881
0·10	0·23318	0·29507	0·35754	0·42034	0·48333	0·54646	0·86300
0·15	0·24478	0·30079	0·35708	0·41355	0·47013	0·52678	0·81059
0·20	0·25819	0·30915	0·36028	0·41151	0·46281	0·51415	0·77124
0·25	0·27104	0·31752	0·36410	0·41074	0·45744	0·50417	0·73809
0·30	0·28248	0·32486	0·36731	0·40981	0·45235	0·49493	0·70803
0·35	0·29216	0·33073	0·36935	0·40803	0·44674	0·48547	0·67939
0·40	0·29997	0·33494	0·36998	0·40506	0·44016	0·47530	0·65123
0·45	0·30589	0·33745	0·36907	0·40072	0·43240	0·46412	0·62294
0·50	0·30996	0·33824	0·36657	0·39494	0·42334	0·45177	0·59419

Names of Materials.	Specific Gravity.	Weight of 1 Cubic Foot in lbs.	Tenacity per Square Inch in lbs.	Crushing Force per Square Inch in lbs.	Modulus of Elasticity E.	Modulus of Rupture S.
Coal (Bonlavooneen) -	1·436 Mt.	89·75				
do. (coke) - - -	1·596 Mt.	99·75				
do. (Corgee) - -	1·403 Mt.	87·68				
do. (coke) - - -	1·656 Mt.	103·50				
do. (Staffordshire) -	1·240	78·12				
do. (Swansea) - -	1·357 K.	84·81				
do. (Wigan) - -	1·268 K.	79·25				
do. (Glasgow) - -	1·290	80·62				
do. (Newcastle) -	1·257 K.	78·56				
do. (common cannel)	1·232 K.	77·00				
do. (slaty cannel) -	1·426 K.	89·12				
Copper (cast) - -	8·607	537·93	19072			
do. (sheet) - -	8·785	549·06				
do. (wire-drawn) -	8·878	560·00	61228			
do. (in bolts) - -	- -	- -	48000			
Crab-tree - - -	0·765	47·80	- -	6499 H.		
Deal (Christiania middle) - - - -	0·698 B.	43·62	12400	- -	1672000 B.	9864 B.
do. (Memel middle) -	0·590 B.	36·87	- -	- -	1535200 B.	10386 B.
do. (Norway spruce) -	0·340	21·25	17600			
do. (English) - -	0·470	29·37	7000			
Earth, (rammed) - -	1·584 Pa.	99·00				
Elder - - - -	0·695 M.	43·43	10230	8467 H.		
Elm (seasoned) - -	0·588 C.	36·75	13489 M.	10331 H.	699840 B.	6078 B.
Fir (New England) -	0·553 B.	34·56	- -	- -	2191200 B.	6612 B.
do. (Riga) - -	0·753 B.	47·06	{ 11549 B. to 12857 B.	5748 H. to 6586 H.	1328800 B. 869600 B.	6648 B. 7572 B.
do. (Mar Forest) -	0·693 B.	43·31				
Flint - - - -	2·630 T.	164·37				
Glass (plate) - - -	2·453	153·31	9420			
Gravel - - - -	1·920	120·00				
Granite (Aberdeen) -	2·625	164·00				
do. (Cornish) - -	2·662	166·30				
do. (red Egyptian) -	2·654	165·80				
Hawthorn - - -	0·91 Be.	38·12	10500 Be.			
Hazel - - - -	0·86 Be.	53·75	18000 Be.			
Holly - - - -	0·76 Be.	47·5	16000 Be.			
Horn of an ox - -	1·689 M.	105·56	8949			
Hornbeam (dry) - -	0·760 R.	47·50	20240 Be.	7289 H.		
Iron (wrought English)	7·700	481·20	25¾ tons, La.			
do. (in bars) - -	{ ·7600 to 7·800	475·50 487·00	25½ tons, La.			
do. (hammered) -	- -	- -	30 tons, Bru.			
do. (Russian), in bars	- -	- -	27 tons, La.			
do. (Swedish) in bars	- -	- -	32 tons, R.			
do. (English) in wire 1-10th inch diameter	} - -	- -	{ 36 to 43 tons, Te.			
do. (Russian) in wire 1-20th to 1-30th inch diameter - - -	} - -	- -	{ 60 to 91 tons, La.			
do. rolled in sheets and cut lengthwise -	- -	- -	14 tons, Mi.			
do. cut crosswise	- -	- -	18 tons, Mi.			
do. in chains, oval links, 6 inches clear, iron ½ inch diameter	- -	- -	21½ tons, Br.			
do. (Brunton's) with stay across link -	- -	- -	25 tons, B.			
Iron, cast (old Park) -	- -	- -	- -	- -	18014400 T.	48940 T.
do. (Adelphi) - -	- -	- -	- -	- -	18353600 T.	45360 T.
do. (Alfreton) - -	- -	- -	- -	- -	17686400 T.	44046 T.
do. (scrap) - - -	- -	- -	- -	- -	18082000 T.	45828 T.
do. (Carron, No. 2. cold blast) - -	7·066 H.	441·62	16683 H.	106375 H.	17270500 H.	38556 H.*
do. (hot blast) -	7·046 H.	440·37	13505 H.	108540 H.	16085000 H.	37503 H.*

* The numbers marked with an asterisk are calculated from the experiments of Messrs. Hodgkinson and Fairbairn.

Grains in 1 lb. ditto	=7000
Grains in a cubic inch of distilled water, Bar. 30 in., Th. 62°	=252·458
Cubic inches in an ounce of water	=1·73298
Cubic inches in the Imperial gallon	=277·276
Feet in a geographical mile	=6075·6
Log. of ditto	=3·7835892
Feet in a statute mile	=5280
Log. of ditto	=3·7226339
Length of seconds' pendulum in inches	=39·19084
Cubic inches in 1 cwt. of cast iron	=430·25
— Bar iron	=397·60
— Cast brass	=368·88
— Cast copper	=352·41
— Cast lead	=272·80
Cubic feet in 1 ton of paving stone	=14·835
— Granite	=13·505
— Marble	=13·070
— Chalk	=12·874
— Limestone	=11·273
— Elm	=64·460
— Honduras mahogany	=64·000
— Mar Forest fir	=51·650
— Beech	=51·494
— Riga fir	=47·762
— Ash and Dantzic oak	=47·158
— Spanish mahogany	=42·066
— English oak	=36·205
To find the weight in lbs. of 1 foot of common rope, multiply the square of its circumference in inches by	·044 to ·046
Ditto for a cable	·027

Note. — The numerical values of the functions of π in this table were calculated by Mr. Goodwin. These, together with the numbers of cubic inches and feet per cwt. or ton of different materials, are taken from the late Dr. Gregory's excellent treatise, entitled *Mechanics for Practical Men*. The other numbers of the table are principally taken from Mr. Babbage's *Tables of Logarithms* and the *Aide Memoire* of M. Morin.

THE END.

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